SAMPLE SOLUTIONS EXERCISE 11

EXERCISE 11.1: EXACT DIFFERENTIAL (8P)

A differential $\mathrm{d}w = a(x,y)\,\mathrm{d}x + b(x,y)\,\mathrm{d}y$ in two dimensions is called *exact* if the integral

$$I = \int_{(x_1,y_1)}^{(x_2,y_2)} \mathrm{d}w$$

is independent of the form of the integration contour between the two points (x_1, y_1) and (x_2, y_2). This allows an exact differential $\mathrm{d}w$ to be associated with a corresponding potential $w(x, y)$ such that $I = w(x_2, y_2) - w(x_1, y_1)$.

(a) Show that an exact differential in 2D obeys the relation (1P)

$$\frac{\partial a(x,y)}{\partial y} = \frac{\partial b(x,y)}{\partial x}$$

(b) Show that the following two differentials are exact: (2P)

$$\mathrm{d}r = (3x^2y + 3y^2)\,\mathrm{d}x + (x^3 + 6xy)\,\mathrm{d}y \quad \text{and} \quad \mathrm{d}s = ((1+x)\,\mathrm{d}x + 2xy\ \mathrm{d}y)e^{x+y^2}$$

(c) Determine the potentials corresponding to the two exact differentials given in (b). (2P)

(d) Let us denote non-exact differentials by $đw$. Now let $đu = a(x,y)\,\mathrm{d}x + b(x,y)\,\mathrm{d}y$ be a non-exact differential. Then we can multiply $đw$ with a function $g(x, y)$, known as the so-called integrating factor, such that $\mathrm{d}v = g(x,y)\,đu$ is an exact differential. Use the result of (a) to derive a partial differential equation for the integrating factor $g(x, y)$. (1P)

(e) Show that $đu = y\,\mathrm{d}x - x\,\mathrm{d}y$ is not exact. Determine the integrating factor $g(x, y)$ by using the ansatz $g(x, y) = \alpha x^\mu y^\nu$ and the partial differential equation from (d), deriving a relation between μ and ν. (2P)

SAMPLE SOLUTION

(a) If $\mathrm{d}w$ is exact, then there will be a potential $w(x, y)$ such that

$$\mathrm{d}w = \underbrace{\frac{\partial w(x,y)}{\partial x}}_{=a(x,y)}\mathrm{d}x + \underbrace{\frac{\partial w(x,y)}{\partial y}}_{=b(x,y)}\mathrm{d}y$$

We consider the following derivatives of the coefficients:

$$\frac{\partial a(x,y)}{\partial y} = \frac{\partial^2 w(x,y)}{\partial y \partial x} \qquad \text{and} \qquad \frac{\partial b(x,y)}{\partial x} = \frac{\partial^2 w(x,y)}{\partial x \partial y}.$$

Since partial derivatives commute, both expressions are equal, completing the proof. (1P)

(b) We use the criterion given in (a):

$$\mathrm{d}r = a(x,y)\,\mathrm{d}x + b(x,y)\,\mathrm{d}y \quad \text{is exact if and only if} \quad \frac{\partial a}{\partial y} = \frac{\partial b}{\partial x}$$

For $\mathrm{d}r$ we compute the partial derivatives (1P)

$$\frac{\partial(3x^2y+3y^2)}{\partial y} = 3x^2+6y\,, \qquad \frac{\partial(x^3+6xy)}{\partial x} = 3x^2+6y\,.$$

Since the expressions on the r.h.s. are equal, the differential $\mathrm{d}r$ is exact. The same can be shown in the second case

$$\frac{\partial(1+x)e^{x+y^2}}{\partial y} = 2(1+x)ye^{x+y^2}\,, \qquad \frac{\partial 2xye^{x+y^2}}{\partial s} = 2(1+x)ye^{x+y^2}\,,$$

meaning that $\mathrm{d}s$ is also exact. (1P)

(c) Since we can choose the integration contour freely, we should take a particularly simple one. In the first case it is natural to integrate along two straight lines from $(0,0) \to (x,0) \to (x,y)$: (1P)

$$r(x,y) = \int_0^x a(x',0)\,\mathrm{d}x' + \int_0^y b(x,y')\,\mathrm{d}y' = 0 + \int_0^y (x^3+6xy')\,\mathrm{d}y' = x^3y+3xy^2$$

In the second case, it turns out to be somewhat simpler to integrate along the contour $(0,0) \to (0,y) \to (x,y)$: (1P)

$$s(x,y) = 0 + \int_0^x 2xye^{x+y^2} = xe^{x+y^2}$$

(d) Suppose that the differential

$$\mathrm{d}v = g(x,y)a(x,y)\,\mathrm{d}x + g(x,y)b(x,y)\,\mathrm{d}y$$

is an exact one. Because of (a) this implies that

$$\frac{\partial[ga]}{\partial y} = \frac{\partial[gb]}{\partial x}$$

Using the product rule we therefore arrive at the following partial differential equation:

$$a\partial_y g + g\partial_y a = b\partial_x g + g\partial_x b$$

(e) The differential $\text{đ}u = y\,\mathrm{d}x - x\,\mathrm{d}y$ corresponds to the special case where $a(x,y) = y$ and $b(x,y) = -x$. First we check the criterion in (a), finding that (1P)

$$\frac{\partial a(x,y)}{\partial y} = 1 \;\neq\; \frac{\partial b(x,y)}{\partial x} = -1.$$

Now we would like to determine the integrating factor, using the ansatz $g = \alpha x^\mu y^\nu$. Inserting this ansatz into the partial differential equation obtained in (d), we obtain the condition (1P)

$$y\alpha x^\mu \nu y^{\nu-1} + \alpha x^\mu y^\nu 1 = -x\alpha\mu x^{\mu-1}y^\nu - \alpha x^\mu y^\nu 1.$$

Here α can be chosen freely, so let us take $\alpha = 1$. For μ, ν a possible choice is $\nu + 1 = -\mu - 1$. The simplest solution is $\mu = \nu = -1$, meaning that $g(x,y) = \frac{1}{xy}$. In fact, the resulting differential is exact: (1P)

$$\mathrm{d}v = \frac{\mathrm{d}x}{x} - \frac{\mathrm{d}y}{y} \quad \Rightarrow \quad v = \ln x - \ln y.$$

$(\Sigma = \mathbf{8P})$