**Sample Solutions Exercise 11**

**Exercise 11.1: Exact differential**

A differential \( dw = a(x, y) \, dx + b(x, y) \, dy \) in two dimensions is called *exact* if the integral

\[
I = \int_{(x_1, y_1)}^{(x_2, y_2)} dw
\]

is independent of the form of the integration contour between the two points \((x_1, y_1)\) and \((x_2, y_2)\). This allows an exact differential \( dw \) to be associated with a corresponding potential \( w(x, y) \) such that \( I = w(x_2, y_2) - w(x_1, y_1) \).

(a) Show that an exact differential in 2D obeys the relation

\[
\frac{\partial a(x, y)}{\partial y} = \frac{\partial b(x, y)}{\partial x}
\]

(b) Show that the following two differentials are exact:

\[
dr = (3x^2y + 3y^2) \, dx + (x^3 + 6xy) \, dy \quad \text{and} \quad ds = ((1 + x) \, dx + 2xy \, dy)e^{x+y^2}
\]

(c) Determine the potentials corresponding to the two exact differentials given in (b).

(d) Let us denote non-exact differentials by \( d\omega \). Now let \( d\omega = a(x, y) \, dx + b(x, y) \, dy \) be a non-exact differential. Then we can multiply \( d\omega \) with a function \( g(x, y) \), known as the so-called integrating factor, such that \( d\nu = g(x, y) \, d\omega \) is an exact differential. Use the result of (a) to derive a partial differential equation for the integrating factor \( g(x, y) \).

(e) Show that \( d\omega = y \, dx - x \, dy \) is not exact. Determine the integrating factor \( g(x, y) \) by using the ansatz \( g(x, y) = \alpha x^\mu y^\nu \) and the partial differential equation from (d), deriving a relation between \( \mu \) and \( \nu \).

**Sample Solution**

(a) If \( dw \) is exact, then there will be a potential \( w(x, y) \) such that

\[
dw = \left( \frac{\partial w(x, y)}{\partial x} \right) dx + \left( \frac{\partial w(x, y)}{\partial y} \right) dy
\]

\[
= a(x, y) \, dx + b(x, y) \, dy
\]

We consider the following derivatives of the coefficients:

\[
\frac{\partial a(x, y)}{\partial y} = \frac{\partial^2 w(x, y)}{\partial y \partial x} \quad \text{and} \quad \frac{\partial b(x, y)}{\partial x} = \frac{\partial^2 w(x, y)}{\partial x \partial y}.
\]

Since partial derivatives commute, both expressions are equal, completing the proof.

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Solutions Sheet 11
(b) We use the criterion given in (a):

\[ dr = a(x, y) \, dx + b(x, y) \, dy \]

is exact if and only if \( \frac{\partial a}{\partial y} = \frac{\partial b}{\partial x} \).

For \( dr \) we compute the partial derivatives

\[ \frac{\partial (3x^2y + 3y^2)}{\partial y} = 3x^2 + 6y, \quad \frac{\partial (x^3 + 6xy)}{\partial x} = 3x^2 + 6y. \]

Since the expressions on the r.h.s. are equal, the differential \( dr \) is exact. The same can be shown in the second case

\[ \frac{\partial (1 + x)e^{x+y^2}}{\partial y} = 2(1 + x)ye^{x+y^2}, \quad \frac{\partial 2xye^{x+y^2}}{\partial x} = 2(1 + x)ye^{x+y^2}, \]

meaning that \( ds \) is also exact.

(c) Since we can choose the integration contour freely, we should take a particularly simple one. In the first case it is natural to integrate along two straight lines from \((0, 0) \to (x, 0) \to (x, y)\):

\[ r(x, y) = \int_0^x a(x', 0) \, dx' + \int_0^y b(x, y') \, dy' = 0 + \int_0^y (x^3 + 6xy') \, dy' = x^2y + 3xy^2 \]

In the second case, it turns out to be somewhat simpler to integrate along the contour \((0, 0) \to (0, y) \to (x, y)\):

\[ s(x, y) = 0 + \int_0^x 2xye^{x+y^2} = xe^{x+y^2} \]

(d) Suppose that the differential

\[ dv = g(x, y)a(x, y) \, dx + g(x, y)b(x, y) \, dy \]

is an exact one. Because of (a) this implies that

\[ \frac{\partial [ga]}{\partial y} = \frac{\partial [gb]}{\partial x} \]

Using the product rule we therefore arrive at the following partial differential equation:

\[ a\partial_y g + g\partial_y a = b\partial_x g + g\partial_x b \]

(e) The differential \( du = y \, dx - x \, dy \) corresponds to the special case where \( a(x, y) = y \) and \( b(x, y) = -x \). First we check the criterion in (a), finding that

\[ \frac{\partial a(x, y)}{\partial y} = 1 \neq \frac{\partial b(x, y)}{\partial x} = -1. \]

Now we would like to determine the integrating factor, using the ansatz \( g = \alpha x^\mu y^{\nu-1} \). Inserting this ansatz into the partial differential equation obtained in (d), we obtain the condition

\[ y\alpha x^{\mu+1}y^{\nu-1} + \alpha x^\mu y^{\nu-1} - x\alpha x^\mu y^{\nu-1} = -x\alpha x^{\mu+1}y^{\nu-1} - \alpha x^\mu y^{\nu-1}. \]

Here \( \alpha \) can be chosen freely, so let us take \( \alpha = 1 \). For \( \mu, \nu \) a possible choice is \( \nu + 1 = -\mu - 1 \). The simplest solution is \( \mu = \nu = -1 \), meaning that \( g(x, y) = \frac{1}{xy} \). In fact, the resulting differential is exact:

\[ dv = \frac{dx}{x} - \frac{dy}{y} \Rightarrow v = \ln x - \ln y. \]
$(\Sigma = 8P)$