Let us consider a single quantum-mechanical particle in a cylinder with a movable piston. The system is modeled as a one-dimensional quantum system with the potential

\[ V(x, t) = \begin{cases} 0 & 0 < x < L(t) \\ \infty & \text{otherwise} \end{cases} \]

The piston is first at rest, then moves with velocity \( v \), and finally stops again:

\[ L(t) = \begin{cases} L_0 & t \leq 0 \\ L_0 + vt & 0 < t < T \\ L_1 = L_0 + vT & t \geq T \end{cases} \]

The particle evolves according to the Schrödinger equation

\[ i\hbar \frac{\partial}{\partial t} \psi(x, t) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi(x, t) \]

subjected to the boundary conditions \( \psi(0, t) = \psi(L(t), t) = 0 \).

(a) Determine the eigenenergies \( E_n \) and eigenfunctions \( \psi_n(x, t) \) for \( t < 0 \). (1P)

(b) Verify that

\[ \phi_n(x, t) = \sqrt{\frac{2}{L(t)}} \exp\left( i\alpha x^2 - \frac{in^2\pi^2}{4\alpha L(t)} \right) \sin\left( \frac{n\pi x}{L(t)} \right) \]

for \( n = 1, 2, \ldots \) and with \( \alpha = \frac{mL_0v}{2\hbar} \) solves the time-dependent Schrödinger equation in the moving phase. (2P)
(c) Prove that for $t = 0$ the non-moving states $\psi_n(x, 0)$ provide an orthonormal basis. Prove the same for the moving states $\phi_n(x, 0)$ as well. (2P)

(d) Consider the moving eigenfunction $\phi_n(x, 0)$ at $t = 0$ and expand it to first order in the velocity $v$. Let us refer to this approximated wave function as $\tilde{\phi}_n(x, 0)$. (2P)

(e) Suppose that for $t < 0$ the particle is in its ground state $\psi_1(x, t)$. When the piston suddenly begins to move, the particle is no longer in its ground state, i.e., the coefficients $c_n$ in the expansion

$$|\psi_1\rangle = \sum_{n=1}^{\infty} \left( \frac{\tilde{\phi}_n}{c_n} \right) |\phi_n\rangle$$

do not vanish for $n > 1$. This means that the particle is now in a nontrivial linear combination of all eigenmodes. To see this, compute the scalar products $\langle \tilde{\phi}_1 | \psi_1 \rangle$ and $\langle \tilde{\phi}_2 | \psi_1 \rangle$ to lowest order in $v$ at $t = 0$. (2P)

(f) In the adiabatic limit of very very small velocities the particle remains practically in its ground state ($c_1 \approx 1$ and $c_n \approx 0$ for $n > 1$). Assuming that the particle is actually in the state $\phi_1$ during expansion, compute the expectation value of the energy as a function of time. (2P)

(g) Calculate the energy loss $\Delta E = \langle E \rangle_{t=0} - \langle E \rangle_{t=T}$ during expansion and use it to calculate the force $F$ that the particle exerts on the piston as well as the mechanical work $\Delta W$ performed by the piston. (1P)

You may use Mathematica® or similar ACSs to solve this exercise.