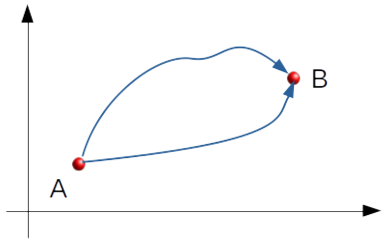

Exact differentials can be integrated independent on the path.

EXERCISE 11.1: EXACT DIFFERENTIAL (8P)

A differential $\mathrm{d}w = a(x,y)\,\mathrm{d}x + b(x,y)\,\mathrm{d}y$ in two dimensions is called *exact* if the integral

$$I = \int_{(x_1,y_1)}^{(x_2,y_2)} \mathrm{d}w$$

is independent of the form of the integration contour between the two points (x_1, y_1) and (x_2, y_2). This allows an exact differential $\mathrm{d}w$ to be associated with a corresponding potential $w(x, y)$ such that $I = w(x_2, y_2) - w(x_1, y_1)$.

(a) Show that an exact differential in 2D obeys the relation (1P)

$$\frac{\partial a(x,y)}{\partial y} = \frac{\partial b(x,y)}{\partial x}$$

(b) Show that the following two differentials are exact: (2P)

$$\mathrm{d}r = (3x^2y + 3y^2)\,\mathrm{d}x + (x^3 + 6xy)\,\mathrm{d}y \quad \text{and} \quad \mathrm{d}s = ((1+x)\,\mathrm{d}x + 2xy\ \mathrm{d}y)e^{x+y^2}$$

(c) Determine the potentials corresponding to the two exact differentials given in (b). (2P)

(d) Let us denote non-exact differentials by $đw$. Now let $đu = a(x,y)\,\mathrm{d}x + b(x,y)\,\mathrm{d}y$ be a non-exact differential. Then we can multiply $đw$ with a function $g(x,y)$, known as the so-called integrating factor, such that $\mathrm{d}v = g(x,y)\,đu$ is an exact differential. Use the result of (a) to derive a partial differential equation for the integrating factor $g(x,y)$. (1P)

(e) Show that $đu = y\,\mathrm{d}x - x\,\mathrm{d}y$ is not exact. Determine the integrating factor $g(x,y)$ by using the ansatz $g(x,y) = \alpha x^\mu y^\nu$ and the partial differential equation from (d), deriving a relation between μ and ν. (2P)

(Σ = 8P)

Because of the delay only 8P this week. To be handed in on Monday, January 13, at the beginning of the lecture or directly to your tutor.