SAMPLE SOLUTIONS EXERCISE 6

EXERCISE 6.1: ENTROPY OF OVERLAPPING PROBABILITY DENSITIES (4P)

Let $p(x)$ be a given continuously differentiable probability density. Let us shift it to the right and to the left by $x \to x \pm a$ and create a mixed probability density of the form

$$q(x) := \frac{1}{2}\Big(p(x-a) + p(x+a)\Big).$$

The purpose of this exercise is that the entropy is minimal for $a = 0$ where both of them match, that is, the entropy is minimized in the case of a perfect overlap.

(a) Show that $\int_{-\infty}^{+\infty} \Big(p'(x+a) - p'(x-a)\Big)\,\mathrm{d}x = 0 \quad \forall a.$ (1P)

(b) Prove that the entropy $H_a = -\int_{-\infty}^{+\infty} q(x) \ln q(x)\,\mathrm{d}x$ is extremal for $a = 0$. (1P)

(c) Show that the extremum at $a = 0$ is a minimum. You can assume that $p(x)$ and all its derivatives vanish at $x = \pm\infty$. (2P)

Note: Is this minimization of overlapping functions useful? Yes, it is, you can use it to tune your piano, see http://piano-tuner.org. Available for Windows, Mac, Android, and iOS.

SAMPLE SOLUTION

(a)

$$\int_{-\infty}^{+\infty} \Big(p'(x+a) - p'(x-a)\Big)\,\mathrm{d}x = \int_{-\infty}^{+\infty} \frac{\mathrm{d}}{\mathrm{d}a}\Big(p(x+a) + p(x-a)\Big)$$

$$= \frac{\mathrm{d}}{\mathrm{d}a}\Big[\int_{-\infty}^{+\infty} p(x+a)\,\mathrm{d}x + \int_{-\infty}^{+\infty} p(x-a)\,\mathrm{d}x\Big] = \frac{\mathrm{d}}{\mathrm{d}a}(1+1) = 0$$

(b) The entropy reads:

$$H_a = -\int_{-\infty}^{+\infty} \frac{1}{2}\Big(p(x-a) + p(x+a)\Big) \ln \frac{1}{2}\Big(p(x-a) + p(x+a)\Big).$$

As a necessary condition for an extremal point, $\frac{\mathrm{d}}{\mathrm{d}a} H_a = 0$ has to vanish. Carrying out the derivative we obtain

$$\frac{\mathrm{d}}{\mathrm{d}a} H_a = \frac{1}{2}\int_{-\infty}^{+\infty} \Big[1 + \ln\Big(\frac{1}{2}(p(x-a) + p(x+a))\Big)\Big] \cdot \Big(p'(x+a) - p'(x-a)\Big) = 0$$

Clearly, the round bracket vanishes for $a = 0$, hence we have an extremum here.

(c) A straight-forward calculation yields

$$\frac{\mathrm{d}^2}{\mathrm{d}a^2} H_a \Big|_{a=0} = -\int_{-\infty}^{+\infty} (1+\ln p(x)) p''(x)\,\mathrm{d}x$$

We integrate this expression by parts:

$$\frac{\mathrm{d}^2}{\mathrm{d}a^2} H_a \Big|_{a=0} = \underbrace{-\Big[(1+\ln p(x))p'(x)\Big]_{-\infty}^{+\infty}}_{=0 \text{ since } p'(\pm\infty)=0} + \int_{-\infty}^{+\infty} \underbrace{\frac{1}{p(x)}}_{\geq 0} \underbrace{p'(x)p'(x)}_{\geq 0}\,\mathrm{d}x > 0$$

Hence we have a minimum.

Exercise 6.2: Markov process on a ladder (4P)

Consider an infinitely high ladder standing on the ground with rungs enumerated by $1, 2, \ldots$. Imagine someone who jumps upwards by one rung with rate w_u and downwards by one rung with rate w_d.

(a) Write down the master equation. (1P)

(b) Specify the matrix elements of the Liouville operator $\mathcal{L}$. (1P)

(c) One of the eigenvalues of $\mathcal{L}$ is zero. Compute the corresponding left and the right eigenvector of $\mathcal{L}$. (1P)

(d) Show that the right eigenvector represents a probability distribution which is normalizable only if $w_u < w_d$. Why? (1P)

Sample Solution

(a) Denoting by $P_i(t)$ the probability to find the walker on rung i at time T, the master equation reads (1P)

$$\frac{\mathrm{d}}{\mathrm{d}t} P_i(t) = P_{i-1}(t) w_u + P_{i+1}(t) w_d - P_i(t)(w_u + w_d)$$

with the "boundary condition" that there are no rungs below $i = 1$, that is $P_0(t) = 0$.

(b) The Liovillian is tridiagonal: (1P)

$$\mathcal{L} = \begin{pmatrix} w_u & -w_d & & & & \\ -w_u & w_d + w_u & -w_d & & & \\ & -w_u & w_d + w_u & -w_d & & \\ & & \cdots & \cdots & \cdots & \\ & & & \cdots & \cdots & \cdots \end{pmatrix}$$

It can also be written as

$$\mathcal{L}_{ij} = -w_d \delta_{i+1,j} - w_u \delta_{i,j+1} + w_u \delta_{i,1}\delta_{j,1} + (w_u + w_d)\delta_{ij}(1 - \delta_{i,1}).$$

(c) The (unnormalized) eigenvectors $|0\rangle$ and $\langle 0|$ corresponding to the eigenvalue 0 can be computed by first fixing the first compotent (e.g. setting it to 1) and then calculating the following components line by line or row by row. The result reads: (1P)

$$|0\rangle = \begin{pmatrix} 1 \\ w_u/w_d \\ w_u^2/w_d^2 \\ w_u^3/w_d^3 \\ \dots \end{pmatrix}, \quad \langle 0| = \langle \Sigma| = (1, 1, 1, 1, \dots).$$

Note: Note that left and right

(d) The normalized right eigenvector reads $|P_0\rangle = \frac{1}{\langle \Sigma|0\rangle}|0\rangle$. Computing the normalization factor

$$\langle \Sigma|0\rangle = \sum_{i=0}^{\infty}(w_u/w_d) = \begin{cases} \text{divergent} & \text{if } w_u \geq w_d \\ \frac{1}{1-w_u/w_d} & \text{if } w_u < w_d. \end{cases}$$

is only defined for $w_u < w_d$. This is because otherwise the walker continues to climb up, never reaching a stationary state.

⇒ Please turn over

Exercise 6.3: Playing with black holes **(4P)**

A (non-rotating electrically neutral) black hole is a space-time singularity described by a single parameter, namely, its mass M. It is surrounded by a spherical event horizon with radius $r = 2GM/c^2$ and surface area $A = 4\pi r^2$, where G is the gravitational constant.[1]

(a) Suppose that a black hole has a certain entropy H depending on M. Compute the black hole temperature T for given $H(M)$ using the Clausius relation $\mathrm{d}E = T\,\mathrm{d}H$ and Einsteins $E = Mc^2$. (1P)

(b) A black hole with temperature $T > 0$ is not black: it is expected to emit radiation like a black body. According to Wien's displacement law we expect a typical wave length $\lambda \approx \hbar c/4k_B T$. Since the black hole has only a single relevant length scale, namely, its horizon radius r, it is natural to conjecture that $\lambda \approx r$. Use this conjecture to compute $T(M)$ and $H(M)$. (1P)

(c) Show that $H(M)$ is proportional to the *surface area* of the black hole. (1P)

(d) Ordinary physical units (m,s,kg) are superfluous since there are natural Planck units. For example, the Planck length and area are given by $l_P = \sqrt{\hbar G/c^3}$ and $A_P = l_P^2$ (see e.g. `https://en.wikipedia.org/wiki/Planck_units`). How many *bits* per unit Planck area reside on the surface of a black hole? (1P)

Sample Solution

The following solution is written using the physics convention $H = k_B \ln|\Omega|$. Other conventions such as the information-theoretic $H = \log_2(|\Omega|)$ differ only by a constant factor.

(a) If we combine

$$E = Mc^2 \quad \Rightarrow \quad \mathrm{d}E = c^2\,\mathrm{d}M$$

with the Clausius relation $\mathrm{d}E = T\,\mathrm{d}H$ we get the relation

$$c^2\,\mathrm{d}M = T\,\mathrm{d}H \quad \Rightarrow \quad T = c^2\left(\frac{\mathrm{d}H(M)}{\mathrm{d}M}\right)^{-1} = \frac{c^2}{H'(M)}$$

(b) Setting $\lambda = r = 2GM/c^2$ we get

$$T(M) = \frac{\hbar c}{4k_B\lambda} = \frac{\hbar c^3}{8k_B GM}.$$

Then we can compute $H'(M)$ and integrate it:

$$H'(M) = \frac{c^2}{T(M)} = \frac{8k_B GM}{\hbar c} \quad \Rightarrow \quad H(M) = \frac{4k_B GM^2}{\hbar c}$$

(c) The black hole has the surface area

$$A = 4\pi r^2 = \frac{16\pi G^2 M^2}{c^4} \quad \Rightarrow \quad M^2 = \frac{c^4 A}{16\pi G^2}$$

[1] This exercise uses simplified assumptions. The results will differ from the correct results obtained by Hawking and Bekenstein by constant factors such as π.

Inserting this back into the result of (b) we can express the entropy H as

$$H = \frac{k_B c^3}{4\pi\hbar G} \propto A.$$

This is Hawking's main result: The entropy of a black hole is proportional to its horizon area.

(d) Since $A_P = \hbar G/c^3$ we can rewrite

$$H = \frac{k_B}{4\pi}\frac{A}{A_P}.$$

Since there are $N = A/a_P$ Planck areas on the surface, the entropy $h = H/N$ per Planck area is

$$h = H/N = \frac{k_B}{4\pi}.$$

This is the entropy in physics units (using k_B and ln). To get the number of bits we simply have to divide by $k_B \ln 2$:

$$h[bits] = \frac{1}{4\pi \ln 2}$$

A full calculation gives the same formulat without π. The main point is that one gets 'of the order of 1' bits per planck area.

($\Sigma = 12$P)