SAMPLE SOLUTIONS EXERCISE 5

EXERCISE 5.1: STATISTICS OF MISPRINTS (2P)

In a printing company, the error probability for a typo is $q = 10^{-6}$. The typos occur completely uncorrelated and the probability is the same for all characters.

(a) Let n be the number of correct characters between two subsequent typos. Compute the probability distribution $P(n)$ in an infinitely long text and check if it is properly normalized. (1P)

(b) Determine the mean value and the variance of this distribution. (1P)

SAMPLE SOLUTION

(a) Suppose we just had a typo. Then the probability to get n correct characters followed by another typo is

$$P(n) = (1-q)^n q.$$

As can be verified easily, this probability distribution is correctly normalized.

(b) To solve this part one needs the geometric series and its moments:

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}\,, \qquad \sum_{n=0}^{\infty} n\, a^n = \frac{a}{(1-a)^2}\,, \qquad \sum_{n=0}^{\infty} n^2 a^n = \frac{a(1+a)}{(1-a)^3}.$$

This gives:

$$\langle n \rangle = \sum_{n=0}^{\infty} nP(n) = \frac{1-q}{q}$$

$$VAR(n) = \langle n^2 \rangle - \langle n \rangle^2 = \sum_{n=0}^{\infty} n^2 P(n) - \frac{(1-q)^2}{q^2} = \frac{1-q}{q^2}.$$

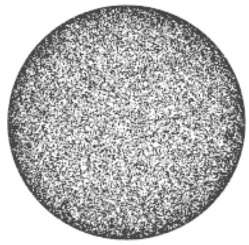
The numerical values are:

$$\langle n \rangle = 999\,999\,, \quad VAR(n) = 999\,999\,000\,000.$$

Thus, the mean value is roughly q^{-1} and the width (square root of variance) is of the same order.

EXERCISE 5.2: UNIFORM PROBABILITY DENSITY ON A UNIT SPHERE (4P)

Consider a unit sphere with standard spherical coordinates $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$. The figure shows uniformly distributed random points on the surface.



(a) Calculate the normalized probability density $p(\theta, \phi)$ which is uniform (i.e. constant per unit area) on the surface of the sphere. (1P)

(b) Show that this probability density factorizes. (1P)

(c) Let X_1 and X_2 be two uncorrelated continuous random variables with a uniform distribution between 0 and 1 (e.g. x_1, x_2 can be drawn from a standard random number generator). Transform $x_1, x_2 \in [0, 1]$ to spherical coordinates θ, ϕ in such a way that the resulting probability on the unit sphere is uniform. (2P)

Sample Solution

(a) The infinitesimal surface element on a sphere reads

$$dS = r^2 \sin\theta \, d\theta \, d\phi$$

A probability density on the unit sphere ($r = 1$) is uniform if $p(\theta, \phi) \, d\theta \, d\phi \propto dS$, therefore

$$p(\theta, \phi) \propto \sin(\theta) \, .$$

Normalizing the probability density by

$$\int_0^\pi d\theta \int_0^{2\pi} d\phi \, p(\theta, \phi) = 1$$

we arrive at

$$p(\theta, \phi) = \frac{1}{4\pi} \sin(\theta).$$

(b) This result shows directly that $p(\theta, \phi) = p(\theta)p(\phi)$ formalally factorizes into the normalized distributions

$$p(\theta) = \frac{1}{2} \sin\theta \, , \qquad p(\phi) = const = \frac{1}{2\pi}.$$

(c) Since $p(\theta, \phi)$ factorizes, the two random variables can be transformed independently, i.e. we are now looking for two function $\theta = f(x_1)$ and $\phi = g(x_2)$ reproducing the probability density calculated above. For the polar angle ϕ this is trivial – we simply have to set

$$\phi = 2\pi z_2$$

As for the azimuthal angle θ, we start with the differential relation

$$p(\theta) \, d\theta \;=\; p(z_1) \, dz_1$$

and integrate it:

$$\int_0^\theta p(\theta') \, d\theta' \;=\; \int_0^{z_1} p(z_1') \, dz_1' \, .$$

Inserting $p(\theta') = \frac{1}{2}\sin\theta'$ and $p(z_1) = 1$ we obtain

$$\frac{1}{2}\int_0^{\theta} \sin\theta'\,\mathrm{d}\theta' = \int_0^{z_1} \mathrm{d}z_1' \qquad \Rightarrow \qquad \frac{1}{2}(1-\cos\theta) = z_1$$

Therefore, the final result reads

$$\boxed{\theta = \arccos(1-2z_1)\,, \quad \phi = 2\pi z_2}$$

Exercise 5.3: Entropy of spin-1 particles **(6P)**

Consider N distinguishable classical spin-1 particles where each of them is in one of three spin states $S_z = 0, \pm 1$. Let n_0, n_+ and n_- (with $N = n_0 + n_+ + n_-$) be the corresponding occupation numbers.

(a) Calculate the entropy for given n_0, n_+, n_- (with $N = n_0 + n_- + n_+$). (1P)

(b) Apply Stirling's formula $\ln(m!) = m\ln m - m$ to derive an approximate expression for the entropy for large N. Determine the maximum of the approximated entropy and the corresponding values of the occupation numbers. (2P)

(c) Why is the approximated entropy determined in (b) too large? (1P)

(d) Show that the improved Stirling formula $\ln(m!) = m\ln m - m + \frac{1}{2}\ln(2\pi m)$ gives in fact an additional correction that decreases the value of the entropy determined in (b). (1P)

(e) Impose the additional constraint $n_- = 2n_+$ and determine the occupation numbers for which the approximated entropy is maximal. (1P)

Sample Solution

(a) For given n_0, n_+, n_- the number of possible configurations is given by the trinomial numbers

$$|\Omega_{n_0,n_+,n_-}| = \binom{N}{n_0, n_+, n_-} = \frac{N!}{n_0!n_+!n_-!}.$$

Hence the entropy is (1P)

$$H(n_0, n_+, n_-) = \ln(N!) - \ln(n_0!) - \ln(n_+!) - \ln(n_-!)$$

(b) Inserting Stirling's formula $\ln(m!) = m\ln m - m$ we get:

$$H(n_0,n_+,n_-) \approx N\ln N - N - n_0\ln n_0 + n_0 - n_+\ln n_+ + n_+ - n_-\ln n_- + n_-$$
$$\approx N\ln N - n_0\ln n_0 - n_+\ln n_+ - n_-\ln n_-$$

The extremum is located at the point where the partial derivative vanish: (1P)

$$\frac{\partial H(N-n_+-n_-, n_+, n_-)}{\partial n_-} = 0\,, \quad \frac{\partial H(N-n_+-n_-, n_+, n_-)}{\partial n_+} = 0.$$

giving the equations

$$\Rightarrow \quad \ln(N - n_+ - n_-) = \ln n_+, \qquad \ln(N - n_+ - n_-) = \ln n_-$$

$$\Rightarrow \quad n_0 = n_+ = n_- = N/3$$

The approximated entropy for these values reads (1P)

$$H \approx N \ln 3.$$

(c) The maximal entropy of the particles without knowledge (n_0, n_+, n_- unconstrained) is $\ln(|\Omega| = \ln(3^n) = N \ln 3$. The approximate entropy *for fixed values* of the occupation numbers (constaints $n_0 = n_+ = n_- = N/3$) takes on the same value as in the unrestricted case, but we expect a constraint to decrease the entropy. (1P)

(d) Repeating the calculation with the improved Stirling formula $\ln(m!) = m \ln m - m + \frac{1}{2}\ln(2\pi m)$ we get (for symmetry reasons) the same location of the extremum ($n_0 = n_+ = n_- = \frac{1}{3}$), but the value of the entropy this point is not $H \approx N \ln 3$ but somewhat lower:

$$H \approx N \ln 3 - \ln(3\pi N)$$

(e) Imposing the additional constraint $n_- = 2n_+$, the extremum is located at the point where the following derivative vanishes:

$$\frac{\partial H(N - 3n_+, n_+, 2n_+)}{\partial n_+} = 0,$$

giving the condition

$$3\ln(N - 3n_+) - 3\ln(n_+) - \ln 4 = 0.$$

$$(N - 3n_+)^3 = 4n_+^3 \quad \Rightarrow \quad n_+ = \frac{N}{3 + 2^{2/3}} \approx 0.218N$$

Remark: The results (b)-(d) are interesting in so far as for very large N, most of the entropy is located at the peak in the 1/3-1/3-1/3 point. This reflects the law of large numbers: Almost all relevant configurations (of order N) are close to point where the entropy is maximal and only a few (of order $\ln N$) are not.

(Σ = 12P)