EXERCISE 4.1: CENTRAL LIMIT THEOREM (CLT) (6P)

Let $X_1, X_2, \ldots, X_N$ be statistically independent and identically distributed random variables with the probability density function

$$p_X(x) = \begin{cases} e^{-x} & \text{if } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

With this exercise we would like to demonstrate the CLT for $Z_N = \sum_{i=1}^{N} X_i$ in the limit $N \to \infty$.

(a) Show that the $N$-fold convolution product of $p(x)$ is given by

$$p_{Z_N}(z) = p_{X}^N(z) = (p_{X} * p_{X} * \cdots * p_{X})_{N \text{ times}}(z) = \frac{z^{N-1}}{(N-1)!} e^{-z} \quad (z \geq 0).$$

(b) Check the norm and compute the mean and the variance of $p_{Z_N}(z)$. (1P)

(c) Standardize the distribution by shifting and rescaling $Z_N$ in such a way that the new random variable $\tilde{Z}_N$ has a probability density $p_{\tilde{Z}_N}(z)$ with unit norm, zero mean, and unit variance (see figure).

(d) Show that the cumulant-generating function $K_{\tilde{Z}}(t)$ of the standardized probability density $p_{\tilde{Z}_N}(z)$ is given by

$$K_{\tilde{Z}}(t) = \frac{1}{2} N \ln N - t \sqrt{N} - N \ln \left( \sqrt{N} - t \right).$$

(e) Compute all cumulants $\kappa_n$ and show that only $\kappa_2$ survives in the limit $N \to \infty$. What does it mean? (2P)

**Sample Solution**

(a) Since $p_{Z_1}(z) = p_X(z)$ the assertion can be proven recursively by showing that $p_{Z_N} * p_X = p_{Z_{N+1}}$. In fact, for $z \geq 0$ we have

$$\left( p_{Z_N} * p_X \right)(z) = \int_{-\infty}^{+\infty} p_{Z_N}(z') p_X(z - z') \, dz' = \frac{z^N}{N!} e^{-z} = p_{Z_{N+1}}(z).$$

(b) The moments are

$$m_n = \int_0^{\infty} dz \, z^n p_{Z_N}(z) = \frac{\Gamma(N+n)}{\Gamma(N)} = \frac{(N+n-1)!}{(N-1)!},$$

giving

$$m_0 = 1, \quad m_1 = N, \quad m_2 = N(N+1), \quad \sigma^2 = \text{Var}(Z_N) = m_2 - m_1^2 = N.$$
(c) First we shift the random variable to the center by replacing $Z_N \rightarrow Z_N^{\text{cent}} = Z_N - N$. For the distribution function this requires to shift its argument $z$ in opposite direction, namely $z \rightarrow z + N$:

$$p_{Z_N^{\text{cent}}}(z) = \begin{cases} \frac{(z+N)^{N-1}}{(N-1)!} e^{-z-N} & \text{for } z \geq -N \\ 0 & \text{otherwise.} \end{cases}$$

Next we have to rescale the random variable by $Z_N^{\text{cent}} \rightarrow \tilde{Z}_N = Z_N^{\text{cent}}/\sigma$ in order to get a new random variable $\tilde{Z}_N$ with unit variance. For the distribution function this means to scale reciprocally by $z \rightarrow z\sigma$. In addition, the total function has to be multiplied by $\sigma$ in order to restore normalization:

$$p_{\tilde{Z}_N}(z) = \begin{cases} \sqrt{N} \frac{(z\sqrt{N}+N)^{N-1}}{(N-1)!} e^{-z\sqrt{N}-N} & \text{for } z \geq -\sqrt{N} \\ 0 & \text{otherwise.} \end{cases}$$

(d) Use e.g. Mathematica® to compute the moment-generating function

$$M_{\tilde{Z}_N}(t) = \langle e^{tz} \rangle = \int_{-\sqrt{N}}^{\infty} dz e^{tz} p_{\tilde{Z}_N}(z) = N^{N-1/2} e^{-\sqrt{N} t} \left( \sqrt{N} - t \right)^{-N}.$$ 

The corresponding cumulant-generating function reads

$$K_{\tilde{Z}_N}(t) = \ln M_{\tilde{Z}_N}(t) = -t\sqrt{N} - N \log \left( \sqrt{N} - t \right) + \frac{1}{2} N \ln(N).$$

(e) In order to compute the cumulants $\kappa_n$ we have to calculate the $n$-th derivative of $K_{\tilde{Z}_N}(t)$ with respect to $t$ and then setting $t = 0$. This yields the following results:

$$K_{\tilde{Z}_N}'(t) = \frac{t\sqrt{N}}{\sqrt{N} - t} \Rightarrow \kappa_1 = 0$$

$$K_{\tilde{Z}_N}''(t) = \frac{\sqrt{N}}{(\sqrt{N} - t)^2} \Rightarrow \kappa_2 = 1$$

$$K_{\tilde{Z}_N}^{(n)}(t) = \frac{N(n-1)!}{(\sqrt{N} - t)^n} \Rightarrow \kappa_n = \frac{(n-1)!}{N^{n/2-1}} \quad (n \geq 2)$$

In the limit $N \rightarrow \infty$ the only cumulant which survives is the variance $\kappa_2 = 1$. Therefore, the stanadized distribution converges to a well-defined distribution with a variance $\sigma^2 = 1$ but no other properties. In the lecture we have shown that this is the normal distribution.

⇒ Please turn over
Exercise 4.2: Reconstruction of a Probability Density

(a) Prove the following statement: If the moment-generating function $M_X(t)$ is analytic, then the corresponding probability density $p(x)$ is given by the inverse Fourier transform

$$p(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} ds \, e^{-isx} M_X(is).$$

(b) Consider a probability distribution with the cumulants

$$\{\kappa_0, \kappa_1, \kappa_2, \kappa_3, \ldots\} = \{0, 0, \frac{3}{2}, 0, -\frac{1}{2}, 0, \frac{1}{3}, 0, -\frac{1}{4}, 0, \frac{1}{5}, 0, -\frac{1}{6}, \ldots\}$$

Compute the generating functions $K(t)$ and $M(t)$.

(c) Use (a) to reconstruct the probability density $p(x)$.

Sample Solution

(a) The MGF is defined as $M(t) = \langle e^{tx} \rangle = \int_{-\infty}^{+\infty} dx \, p(x) e^{tx}$, where $t \in \mathbb{R}$. If $M(t)$ was analytic, this would mean that the defining relation is valid everywhere in the complex plane, in particular on the imaginary line, i.e.

$$M_X(is) = \langle e^{isx} \rangle = \int_{-\infty}^{+\infty} dx \, p(x) e^{isx},$$

where $s \in \mathbb{R}$. This implies that $M(is)$ is (up to a possible prefactor) the Fourier transform of the probability distribution. Thus we can invert the above relation by

$$p(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} ds \, e^{-isx} M_X(is).$$

(b) Obviously the nonzero cumulants are even and given by

$$\kappa_{2m} = \begin{cases} \frac{1}{2} + \frac{(-1)^{m+1} m}{m+1} & \text{if } m = 1 \\ \frac{(-1)^{m+1} m}{m+1} & \text{if } m = 2, 3, \ldots \end{cases}$$

Hence the CGF is given by

$$K(t) = \sum_{n=0}^{\infty} \frac{\kappa_n t^n}{n!} = \frac{1}{2} t^2 + \sum_{m=1}^{\infty} \frac{(-1)^{m+1} m!}{m} t^{2m}$$

Since the infinite sum is of the form $\sum_{k=1}^{\infty} \frac{a^k}{k} = -\ln(1 - a)$ we end up with

$$K(t) = \frac{t^2}{2} + \ln(1 + t^2),$$

implying that

$$M(t) = \exp(K(t)) = e^{\frac{1}{2} t^2} (1 + t^2).$$
(c) Using the formula from (a) the probability density \( p(x) \) reads

\[
p(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} ds \, e^{-ixs} M(is) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} ds \, e^{-ixs} e^{-\frac{1}{2}s^2}(1 - s^2)
\]

We can evaluate this expression with Mathematica®, or we can do it by hand as follows: We can express the \(-s^2\) contribution in the integrand by a derivative:

\[
p(x) = \left(1 + \frac{\partial^2}{\partial x^2}\right) \frac{1}{2\pi} \int_{-\infty}^{+\infty} ds \, e^{-ixs} e^{-\frac{1}{2}s^2}.
\]

Using standard methods of quadratic completion the integral equals

\[
\frac{1}{2\pi} \int_{-\infty}^{+\infty} ds \, e^{-ixs} e^{-\frac{1}{2}s^2} = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.
\]

Taking the derivatives we end up with the final result (1P)

\[
p(x) = \frac{x^2 e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}.
\]

**Exercise 4.3: Transformation of probability densities**

Let \( X \) and \( Y \) be two uncorrelated random variables which are both distributed according to a normal distribution with zero mean and unit variance. What is the probability density of the random variable \( Z := X/Y \)?

**Sample Solution**

According to the lecture notes we can compute the new probability density by (1P)

\[
p(z) = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \, p(x) \, p(y) \, \delta(z - \frac{x}{y})
\]

Now we insert \( p(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \) and \( p(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} \) and use the fundamental relation \( \delta(ax) = \frac{1}{|a|} \delta(x) \), giving

\[
p(z) = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \, e^{-\frac{1}{2}(x^2+y^2)} \frac{|y|}{y} \delta(zy - x)
\]

\[
= \int_{-\infty}^{+\infty} dy \, e^{-\frac{1}{2}(z^2+1)y^2} |y| = 2 \int_{0}^{\infty} dy \, e^{-\frac{1}{2}(z^2+1)y^2} y.
\]

Thus we arrive at the result (1P)

\[
p(z) = \frac{1}{\pi(1 + z^2)}.
\]