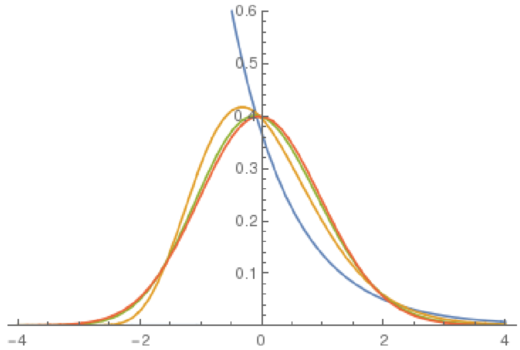
Sample Solutions Exercise 4

Exercise 4.1: Central Limit Theorem (CLT) (6P)

Let $X_1, X_2, \ldots, X_N$ be statistically independent and identically distributed random variables with the probability density function

$$p_X(x) = \begin{cases} e^{-x} & \text{if } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

With this exercise we would like to demonstrate the CLT for $Z_N = \sum_{i=1}^{N} X_i$ in the limit $N \to \infty$.



(a) Show that the N-fold convolution product of $p(x)$ is given by (1P)

$$p_{Z_N}(z) \;=\; p_X^{*N}(z) \;=\; \big(\underbrace{p_X * p_X * \cdots * p_X}_{N\text{ times}}\big)(z) \;=\; \frac{z^{N-1}}{(N-1)!}\,e^{-z} \qquad (z \geq 0)\,.$$

(b) Check the norm and compute the mean and the variance of $p_{Z_N}(z)$. (1P)

(c) Standardize the distribution by shifting and rescaling Z_N in such a way that the new random variable $\tilde{Z}_N$ has a probability density $p_{\tilde{Z}_N}(z)$ with unit norm, zero mean, and unit variance (see figure). (1P)

(d) Show that the cumulant-generating function $K_{\tilde{Z}}(t)$ of the standardized probability density $p_{\tilde{Z}_N}(z)$ is given by (1P)

$$K_{\tilde{Z}}(t) \;=\; \frac{1}{2}N\ln N - t\sqrt{N} - N\ln\left(\sqrt{N} - t\right)\,.$$

(e) Compute all cumulants κ_n and show that only κ_2 survives in the limit $N \to \infty$. What does it mean? (2P)

Sample Solution

(a) Since $p_{Z_1}(z) = p_X(z)$ the assertion can be proven recursively by showing that $p_{Z_N} * p_X = p_{Z_{N+1}}$. In fact, for $z \geq 0$ we have

$$\left(p_{Z_N} * p_X\right)(z) = \int_{-\infty}^{+\infty} p_{Z_N}(z')p_X(z-z')\,\mathrm{d}z' \;=\; \frac{z^N}{N!}e^{-z} \;=\; p_{Z_{N+1}}(z)\,.$$

(b) The moments are

$$m_n = \int_0^\infty \mathrm{d}z\, z^n\, p_{Z_N}(z) = \frac{\Gamma(N+n)}{\Gamma(N)} = \frac{(N+n-1)!}{(N-1)!}\,,$$

giving

$$m_0 = 1, \quad m_1 = N, \quad m_2 = N(N+1), \quad \sigma^2 = \mathrm{Var}(Z_N) = m_2 - m_1^2 = N\,.$$

(c) First we shift the random variable to the center by replacing $Z_N \to Z_N^{cent} = Z_N - N$. For the distribution function this requires to shift its argument z in opposite direction, namely $z \to z + N$:

$$p_{Z_N^{cent}}(z) \;=\; \begin{cases} \frac{(z+N)^{N-1}}{(N-1)!}\, e^{-z-N} & \text{for } z \geq -N \\ 0 & \text{otherwise.} \end{cases}$$

Next we have to rescale the random variable by $Z_N^{cent} \to \tilde{Z}_N = Z_N^{cent}/\sigma$ in order to get a new random variable $\tilde{Z}_N$ with unit variance. For the distribution function this means to scale reciprocally by $z \to z\sigma$. In addition, the total function has to be multiplied by σ in order to restore normalization:

$$p_{\tilde{Z}_N}(z) \;=\; \begin{cases} \sqrt{N}\frac{(z\sqrt{N}+N)^{N-1}}{(N-1)!}\, e^{-z\sqrt{N}-N} & \text{for } z \geq -\sqrt{N} \\ 0 & \text{otherwise.} \end{cases}$$

(d) Use e.g. *Mathematica®* to compute the moment-generating function

$$M_{\tilde{Z}_N}(t) \;=\; \langle e^{tz} \rangle \;=\; \int_{-\sqrt{N}}^{\infty} \mathrm{d}z\, e^{tz}\, p_{\tilde{Z}_N}(z) \;=\; N^{\frac{N-1}{2}+\frac{1}{2}} e^{-\sqrt{N}t} \left(\sqrt{N}-t\right)^{-N}.$$

The corresponding cumulant-generating function reads

$$K_{\tilde{Z}_N}(t) \;=\; \ln M_{\tilde{Z}_N}(t) \;=\; -t\sqrt{N} - N \log\left(\sqrt{N}-t\right) + \frac{1}{2} N \ln(N)\,.$$

(e) In order to compute the cumulants κ_n we have to calculate the n-th derivative of $K_{\tilde{Z}_N}(t)$ with respect to t and then setting $t = 0$. This yields the following results:

$$\begin{aligned} K'_{\tilde{Z}_N}(t) &= \frac{t\sqrt{N}}{\sqrt{N}-t} \quad &\Rightarrow\quad & \kappa_1 = 0 \\ K''_{\tilde{Z}_N}(t) &= \frac{\sqrt{N}}{(\sqrt{N}-t)^2} \quad &\Rightarrow\quad & \kappa_2 = 1 \\ K^{(n)}_{\tilde{Z}_N}(t) &= \frac{N(n-1)!}{(\sqrt{N}-t)^n} \quad &\Rightarrow\quad & \kappa_n = \frac{(n-1)!}{N^{n/2-1}} \quad (n \geq 2) \end{aligned}$$

In the limit $N \to \infty$ the only cumulant which survives is the variance $\kappa_2 = 1$. Therefore, the stanardized distribution converges to a well-defined distribution with a variance $\sigma^2 = 1$ but no other properties. In the lecture we have shown that this is the normal distribution.

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Exercise 4.2: Reconstruction of a probability density (4P)

(a) Prove the following statement: If the moment-generating function $M_X(t)$ is analytic, then the corresponding probability density $p(x)$ is given by the inverse Fourier transform (1P)

$$p(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}s\, e^{-ixs} M_X(is).$$

(b) Consider a probability distribution with the cumulants

$$\{\kappa_0,\, \kappa_1,\, \kappa_2,\, \kappa_3,\, \ldots\} = \{0,\, 0,\, \frac{3}{2},\, 0,\, -\frac{1}{2},\, 0,\, \frac{1}{3},\, 0,\, -\frac{1}{4},\, 0,\, \frac{1}{5},\, 0,\, -\frac{1}{6}, \ldots\}$$

Compute the generating functions $K(t)$ and $M(t)$. (2P)

(c) Use (a) to reconstruct the probability density $p(x)$. (1P)

Sample Solution

(a) The MGF is defined as $M(t) = \langle e^{tx}\rangle = \int_{-\infty}^{+\infty} \mathrm{d}x\, p(x)e^{tx}$, where $t \in \mathbb{R}$. If $M(t)$ was analytic, this would mean that the defining relation is valid everywhere in the complex plane, in particular on the imaginary line, i.e.

$$M_X(is) = \langle e^{isx}\rangle = \int_{-\infty}^{+\infty} \mathrm{d}x\, p(x)e^{isx}\,,$$

where $s \in \mathbb{R}$. This implies that $M(is)$ is (up to a possible prefactor) the Fourier transform of the probability distribution. Thus we can invert the above relation by (1P)

$$p(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}s\, e^{-ixs} M_X(is).$$

(b) Obviously the nonzero cumulants are even and given by

$$\kappa_{2m} = \begin{cases} \frac{1}{2} + \frac{(-1)^{m+1}}{m} & \text{if } m = 1 \\ \frac{(-1)^{m+1}}{m} & \text{if } m = 2, 3, \ldots \end{cases}.$$

Hence the CGF is given by (1P)

$$K(t) = \sum_{n=0}^{\infty} \frac{\kappa_n t^n}{n!} = \tfrac{1}{2}t^2 + \sum_{m=1}^{\infty} \frac{(-1)^{m+1} t^{2m}}{m}$$

Since the infinite sum is of the form $\sum_{k=1}^{\infty} \frac{a^k}{k} = -\ln(1-a)$ we end up with

$$K(t) = \frac{t^2}{2} + \ln(1+t^2)\,,$$

implying that (1P)

$$M(t) = \exp\big(K(t)\big) = e^{\frac{1}{2}t^2}(1+t^2).$$

(c) Using the formula from (a) the probability density $p(x)$ reads

$$p(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}s\, e^{-ixs} M(is) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}s\, e^{-ixs} e^{-\frac{1}{2}s^2}(1-s^2)$$

We can evaluate this expression with *Mathematica*® , or we can do it by hand as follows: We can express the $-s^2$ contribution in the integrand by a derivative:

$$p(x) = \Big(1 + \frac{\partial^2}{\partial x^2}\Big)\frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}s\, e^{-ixs} e^{-\frac{1}{2}s^2}.$$

Using standard methods of quadratic completion the integral equals

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}s\, e^{-ixs} e^{-\frac{1}{2}s^2} = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

Taking the derivatives we end up with the final result (1P)

$$p(x) = \frac{x^2 e^{-\frac{x^2}{2}} I}{\sqrt{2\pi}}.$$

Exercise 4.3: Transformation of probability densities (2P)

Let X and Y be two uncorrelated random variables which are both distributed according to a normal distribution with zero mean and unit variance. What is the probability density of the random variable $Z := X/Y$?

Sample Solution

According to the lecture notes we can compute the new probability density by (1P)

$$p(z) = \int_{-\infty}^{+\infty} \mathrm{d}x \int_{-\infty}^{+\infty} \mathrm{d}y\, p(x)\, p(y)\, \delta(z - \frac{x}{y})$$

Now we insert $p(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$ and $p(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}$ and use the fundamental realtion $\delta(ax) = \frac{1}{|a|}\delta(x)$, giving

$$p(z) = \int_{-\infty}^{+\infty} \mathrm{d}x \int_{-\infty}^{+\infty} \mathrm{d}y\, e^{-\frac{1}{2}(x^2+y^2)}\, |y|\, \delta(zy - x)$$

$$= \int_{-\infty}^{+\infty} \mathrm{d}y\, e^{-\frac{1}{2}(z^2+1)y^2}\, |y| = 2\int_{0}^{\infty} \mathrm{d}y\, e^{-\frac{1}{2}(z^2+1)y^2}\, y\,.$$

Thus we arrive at the result (1P)

$$p(z) = \frac{1}{\pi(1+z^2)}\,.$$

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