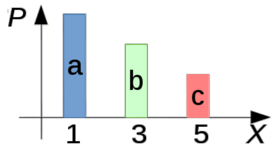
Sample Solutions Exercise 3

Exercise 3.1: Reconstruct probabilities from given moments (4P)

Consider a system with three configurations $\Omega = \{a, b, c\}$ together with an associated map $X : \Omega \to \mathbb{R}$ defined by



$$X_a = 1, \quad X_b = 3, \quad X_c = 5\,.$$

(a) How many moments do you have to know in order to be able to reconstruct the probabilitiy distribution $\{P_a, P_b, P_c\}$ and why?

(b) Suppose that M_1 and M_2 are given (with the map X defined above). Find the solution for the probabilities P_a, P_b, and P_c.

(c) Not all possible values for M_1 are M_2 lead to consistent results. Give a counter-example.

(d) Let $M_1 = \frac{7}{2}$ and $M_2 = 15$. Compute the probabilities P_a, P_b, P_c and find a general formula for the higher moments M_n with $n = 3, 4, \dots \infty$.

Sample Solution

(a) Two! We have three unknowns $\{P_a, P_b, P_c\}$ and therefore we need three equations in total. One of them is the normalization condition $P_a + P_b + P_c = 1$. Therefore, we have to know two moments which provide remaining two equations.

(b) Solving the equations[1]

$$\begin{aligned} 1 &= P_a + P_b + P_c \\ M_1 &= P_a X_a + P_b X_b + P_c X_c \\ M_2 &= P_a X_a^2 + P_b X_b^2 + P_c X_c^2 \end{aligned}$$

we find the solution

$$\begin{aligned} P_a &= \frac{M_2 + X_b X_c - M_1(X_b + X_c)}{(X_a - X_b)(X_b - X_c)} &= \frac{1}{8}(15 - 8M_1 + M_2) \\ P_b &= \frac{M_2 + X_a X_c - M_1(X_a + X_c)}{(X_a - X_b)(X_c - X_b)} &= \frac{1}{4}(-5 + 6M_1 - M_2) \\ P_c &= \frac{M_2 + X_a X_b - M_1(X_a + X_b)}{(X_a - X_c)(X_b - X_c)} &= \frac{1}{8}(3 - 4M_1 + M_2) \end{aligned}$$

(c) Probabilites must not be negative or larger than 1. A possible conterexample is e.g. $M_1 = 1, M_2 = 2$ where $P_b = -1/4$. There are many more examples of that kind.

[1] We encourage the use of algebraic software packages like *Mathematica*® . But please do not submit uncommented notebooks as solutions.

(d) Inserting the solution into $M_n = P_a X_a^n + P_b X_b^n + P_c X_c^n$ we get

$$M_n = \frac{1}{4}(6M_1 - M_2 - 5)3^n + \frac{1}{8}(-4M_1 + M_2 + 3)5^n + \frac{1}{8}(-8M_1 + M_2 + 15)$$

For $n = 1, 2$ this expression reduces correctly to M_1, M_2.

Note: The purpose is to make the students aware of the fact that knowing all moments is like knowing the whole probability distribution (up to subtle mathematical exceptions).

Exercise 3.2: Poisson distribution (6P)

The poisson distribution $P_\lambda(k) = \frac{\lambda^k}{k!} e^{-\lambda}$ can be understood as the limit of the binomial distribution in the case of "rare events".

(a) Let $p = \lambda/N$ and take $N \to \infty$ while keeping λ and k constant. Show that in this limit we can approximate $(1-p)^{N-k} \approx e^{-\lambda}$. (1P)

(b) Show similarly that $\binom{N}{k} \approx \frac{N^k}{k!}$. (1P)

(c) Use (a) and (b) to show that in this limit the binomial distribution tends to the Poisson distribution. (1P)

(d) Check that the Poisson distribution is properly normalized. (1P)

(e) Compute the moment- and cumulant-generating functions. (1P)

(f) Determine all cumulants. (1P)

Sample Solution

(a) Let us consider the logarithm of the left-hand side $(N-k)\ln(1-p)$. Since $p \to 0$ for $N \to \infty$ we can approximate

$$\ln(1-p) \approx -p + \mathcal{O}(p^2).$$

Therefore $(N-k)\ln(1-p) \approx -Np + kp \approx -Np = -\lambda$, hence $(1-p)^{N-k} \approx e^{-\lambda}$

(b) For this we have to show that $N!/(N-k)! \approx N^k$. The take the logarithm and apply Stirlings formula

$$\ln N! - \ln(N-k)! \ \approx \ N \ln N - N - (N-k)\ln(N-k) + (n-k)$$

Since $k \ll N$ we can further approximate $\ln(N-k) \approx \ln N - k/N$. Inserting this approximation in the formula given above and simplifying the expression we arrive at

$$\ln N! - \ln(N-k)! \approx k \ln N.$$

Exponentiating this we finally obtain $\binom{N}{k} \approx \frac{N^k}{k!}$.

(c) Simply insert the results from (a) and (b):

$$\binom{N}{k} p^k (1-p)^{(N-k)} \approx \frac{N^k}{k!} p^k e^{-\lambda} = \frac{\lambda^k e^{-\lambda}}{k!}$$

(d) Check normalization:

$$\sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{-\lambda} e^{+\lambda} = 1.$$

(e) The moment-generating function can also be found in the lecture notes:

$$M(t) = \langle e^{kt} \rangle = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} e^{kt} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^t)^k}{k!} = e^{\lambda(e^t - 1)}.$$

$$K(t) = \ln M(t) = \lambda(e^t - 1).$$

(f) Take the derivative of the cumulant-generating function:

$$\kappa_n = \frac{\mathrm{d}^n}{\mathrm{d}t^n} K(t)\Big|_{t=0} = \lambda e^t\Big|_{t=0} = \lambda.$$

Therefore, the Poisson distribution is special in so far as all cumulants coincide.

Exercise 3.3: Transformation of a probability density (2P)

Suppose that the random variable $X \in [0, 2]$ is distributed according to the probability density $p_X(x) = \frac{3}{4} x(2 - x)$. How is the random variable $Y = X^2$ distributed? Check the normalization of $p_Y(y)$.

Sample Solution

For completeness we first check the normalization of the given distriution (not required):

$$\frac{3}{4} \int_0^2 x(2 - x)\,\mathrm{d}x = 1.$$

The new random variable $Y = X^2$ is distributed according to (see Lecture Notes)

$$p_Y(y) = \frac{p_X(f^{-1}(y))}{|f'(f^{-1}(y))|}.$$

Inserting $p_X(x) = \frac{3}{4} x(2 - x)$ and $y = f(x) = x^2 \quad \Rightarrow \quad x = f^{-1}(y) = \sqrt{y}$ we get (1P)

$$p_Y(y) = \frac{3}{8}\Big(2 - \sqrt{y}\Big).$$

The new domain of Y is no longer $[0, 2]$ but $[0, 4]$. The normalization is (1P)

$$\int_0^4 p_Y(y)\,\mathrm{d}y = \frac{3}{8} \int_0^4 (2 - \sqrt{y}) = 1.$$

(Σ = 12P)