Sample Solutions Exercise 2

Exercise 2.1: Fixed points of a differential equation (3P)

(a) Determine the fixed points of the differential equation $\dot{x}(t) = ax(t)\big(1 - x^2(t)\big)$ and their stability. (2P)

(b) Let $|x(0)| \ll 1$ be very small. Compute the typical time that the system needs to 'switch' to a stable fixed point. (1P)

Sample Solution

(a) The fixed points are given by $\dot{x} = 0$, hence we get three fixed points $x^* = 0$ and $x^* = \pm 1$. (1P) For the stability we compute $\dot{x}$ in the vicinity of the fixed point $x^* + \epsilon$:

$$\dot{x} = a(x^* + \epsilon)\big(1 - (x^* + \epsilon)^2\big) = \underbrace{x^*(1 - (x^*)^2)}_{=0} + \epsilon \underbrace{a\,(1 - 3(x^*)^2)}_{=\gamma} + \mathcal{O}(\epsilon^2)$$

or, in short, $\dot{x} \approx \gamma\epsilon$. A fixed point x^* is stable if $\gamma < 0$. Hence for $a > 0$ both fixed points $x^* = \pm 1$ are stable while $x^* = 0$ is unstable. For $a < 0$ it is the other way round. (1P)

(b) We could solve the differential equation, obtaining the solution

$$x(t) = \pm \frac{1}{\sqrt{1 - Ce^{-2at}}}$$

where C is an integration constant determined by the initial condition:

$$x(0) = \pm \frac{1}{\sqrt{1 - C}} \quad \Rightarrow \quad C = \frac{1}{x(0)^2} - 1.$$

The typical switching time t_f is reached when $x(t)$ made $1/2$ of its way towards the stable fixed point, i.e. $x(t_f) = \pm\frac{1}{2}$:

$$t_f = \frac{\log\left(\frac{1}{3}\left(\frac{1}{\text{x0}^2} - 1\right)\right)}{2a} \approx const - \frac{\ln x(0)}{a}.$$

(1P)

Correction advice: In the result we would like to see $t_f \propto \ln(1/x(0))$ and $t_f \propto 1/a$ with the correct sign. The prefector and the definition 'where' exactly the flip takes place is irrelevant.

Exercise 2.2: Probability density of the logistic map **(6P)**

Let us consider the logistic map

$$x_{n+1} = f(x_n) \qquad \text{with} \qquad f(x) = 4x(1-x)$$

The purpose of this exercise is to show that this map is fully chaotic and that the probability to find x in a certain infinitesimal interval can be computed analytically.

(a) Prove that the Dirac delta function obeys the relation $\delta(ax) = \frac{1}{|a|}\delta(x)$. (2P)

(b) Likewise prove that $\delta(F(x)) = \sum_i \frac{\delta(x-x_i)}{|F'(x_i)|}$, where the x_i are the (non-degenerate) zeros of the differentiable function F, i.e., $F(x_i) = 0$. (1P)

(c) Let $p_n(x_n)$ be the probability density of x_n, which means that $p_n(x_n)\,\mathrm{d}x_n$ is the probability to find x_n in the infinitesimal interval $[x_n, x_n + \mathrm{d}x_n]$. Later in this lecture we will show that probability density one step later is given by

$$p_{n+1}(x_{n+1}) = \int_{-\infty}^{+\infty} \mathrm{d}x_n\; \delta\Big(x_{n+1} - f(x_n)\Big)\, p_n(x_n).$$

Insert $f(x)$ into this expression and use (b) to simplify it. (2P)

(d) A probability density of a map is called *stationary* if it does not change under iteration, i.e. $p_{n+1}(x) = p_n(x)$. Show that the stationary probability density

$$p_n(x) \propto \frac{1}{\sqrt{x(1-x)}}$$

is a stationary solution. (1P)

Sample Solution

(a) The dirac delta function is a distribution that has to be placed into an integral together with a test function. The relation can be proven by substituting $|a|x = y$, implying that $\mathrm{d}x = \mathrm{d}y/|a|$: (1P)

$$\int_{-\infty}^{+\infty} \mathrm{d}x\, \delta(ax) g(x) = \int_{-\infty}^{+\infty} \mathrm{d}x\, \delta(|a|x) g(x) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}y}{|a|} \delta(y) g(y/|a|) = \frac{g(0)}{|a|}.$$

On the other hand

$$\int_{-\infty}^{+\infty} \mathrm{d}x\, \frac{\delta(x)}{|a|} g(x) = \frac{g(0)}{|a|}$$

gives the same result for any test function $g(x)$, hence we can identify the red-colored parts of the integrands. (1P)

(b) The function F is assumed to have non-degenerate zeros x_i, that is, $F(x_i) = 0$ and $F'(x_i) \neq 0$ (the x-axis is crossed but it is not touched). Clearly, $\delta(F(x))$ is non-zero only at the roots of F, i.e., at the points x_i where $F(x_i) = 0$. Since F is differentiable, we may thus Taylor-expand $F(x) = F(x_i) + F'(x_i)(x - x_i) + \mathcal{O}((x-x_i)^2)$ in the vicinity of the roots. Obviously, for each root x_i the prefactor $F'(x_i)$ play the same role as the proportionality constant a in part (a) of this exercise. Finally we have to add over all roots, giving $\delta(F(x)) = \sum_i \frac{\delta(x_i)}{|F'(x_i)|}$. (1P)

(c) We rewrite the equation as

$$p_{n+1}(x_{n+1}) = \int_{-\infty}^{+\infty} \mathrm{d}x_n \, \delta\Big(F(x_n)\Big) \, p_n(x_n),$$

where $F(x_n) := x_{n+1} - f(x_n) = x_{n+1} - 4x_n(1-x_n)$. This function has two roots $F(x_a) = F(x_b) = 0$, namely (1P)

$$x_{a,b} = \frac{1}{2}(1 \pm \sqrt{1 - x_{n+1}}).$$

In both cases the absolute value of the first derivative is given by

$$|F'(x_{a,b})| \;=\; 4\sqrt{1 - x_{n+1}}.$$

Inserting these expressions we get (1P)

$$p_{n+1}(x_{n+1}) = \frac{p(x_a) + p(x_b)}{4\sqrt{1 - x_{n+1}}}.$$

Note: This is a nonlocal map from the function p_n to p_{n+1}. Usually such non-local maps are very difficult to solve analytically. However, as we will see in part (d), there is a simple analytic solution for the stationary solution.

(d) We have to show that

$$p_n(x) = \frac{C}{\sqrt{x(1-x)}} \stackrel{?}{=} p_{n+1}(x),$$

where C is some proportionality constant. With $x_{a,b} = \frac{1}{2}(1 \pm \sqrt{1-x})$ and using (c) we find (1P)

$$p_{n+1}(x) \;=\; \frac{p(x_a) + p(x_b)}{4\sqrt{1-x}} \;=\; \frac{2C/\sqrt{x} + 2C/\sqrt{x}}{4\sqrt{1-x}} \;\stackrel{!}{=}\; p(x). \qquad \square$$

Therefore, this probability distribution is stationary under iteration.

Exercise 2.3: Drunk Driver Statistics (3P)

According to German police statistics in 2009, the age of the drunk drivers involved in car accidents with physical injuries are distributed as follows:

Age	18-24	25-34	35-44	45-54	55-64	≥65
%	26.1	22.4	18.9	18.5	8.7	5.4

At a *given* age, the percentage of *female* drivers involved in such accidents is fairly low:

Age	18-24	25-34	35-44	45-54	55-64	≥65
%	9.1	11.9	15.2	14.4	12.3	9.2

(a) What is probability of being female in an such an accident? (1P)

(b) Compute the probability depending on the age under the condition that the driver is female. Where is the maximum? (2P)

Sample Solution

(a) For simplicity, let 1,2,3,4,5,6 be the six age groups and let us denote male and female drivers by M and W. The exercise specifies the probabilities

$$p(1) = 0.261\,, \quad p(2) = 0.224\,, \quad p(3) = 0.189\,,$$

$$p(4) = 0.185\,, \quad p(5) = 0.087\,, \quad p(6) = 0.054\,,$$

as well as the conditional probabilities

$$p(W|1) = 0.091\,, \quad p(W|2) = 0.119\,, \quad p(W|3) = 0.152\,,$$

$$p(W|4) = 0.144\,, \quad p(W|5) = 0.123\,, \quad p(W|6) = 0.092\,,$$

Then

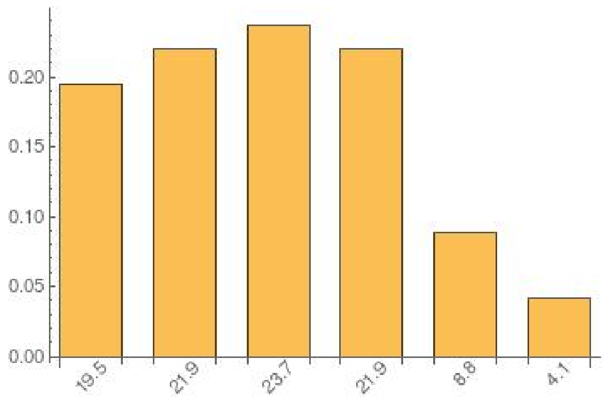
$$p(W) = \sum_{a=1}^{6} p(W|a)p(a) = 0.121,$$

so only 12% of drunk drivers are female. (1P)

(b) To see the age distribution among drunk female drivers we apply Bayes formula: (1P)

$$p(a|W) = \frac{P(w|a)p(a)}{p(W)}\,,$$

getting (1P)



Unlike male drivers, who are at particular risk in the early 20's, female drivers have a maximum at age 35-44.

(Σ = 12P)