Exercise 1.1: Attractor of a quadratic map \( (6P) \)

Let us consider the nonlinear map \( x_{n+1} = f(x_n) \) with \( f(x) = a - x^2 \), where \( a > 0 \) is a constant.

(a) Determine the real-valued fixed points \( x^* = f(x^*) \) and assess their stability. \( (2P) \)

(b) Set \( a := 1 \) and \( x_0 := 0.5 \) and iterate the map numerically up to \( n = 20 \). What happens for \( n \to \infty \) and how do you interpret the result? \( (2P) \)

(c) Prove your observation in (b) analytically. \( (2P) \)

Sample Solution

(a) Solving \( x^* = a - (x^*)^2 \) we find (for \( a > 0 \)) two real-valued the fixed points \( (1P) \)

\[ x^*_\pm = \frac{1}{2} \left( \pm \sqrt{1 + 4a} - 1 \right). \]

In order to check the stability we compute \( (1P) \)

\[ f'(x^*_\pm) = 2x^*_\pm = \pm \sqrt{1 + 4a} - 1. \]

A fixed point is stable if \( |f'(x)| < 1 \). Therefore \( x^*_+ \) is stable for \( a < \frac{3}{4} \), marginal at \( a = \frac{3}{4} \), and unstable for \( a > \frac{3}{4} \). The other fixed point \( x^- \) is always unstable. \( (1P) \)

(b) For \( a = 1 \) and \( x_0 = 0.5 \) the iteration sequence reads (can be computed even with a pocket calculator): \( 0.5 \mid 0.75 \mid 0.4375 \mid 0.808594 \mid 0.346176 \mid 0.880162 \mid 0.225315 \mid 0.949233 \mid 0.990208 \mid 0.0194888 \mid 0.99962 \mid 0.0075948 \mid 0.999999 \mid 1.15362 \times 10^{-6} \mid 1. \mid 2.66165 \times 10^{-12} \mid 1. \mid 0. \mid 1. \mid 0. \mid ... \) \( (1P) \)

That is, until sufficiently many iterations, the sequence effectively alternates between 0 and 1. This phenomenon is called period doubling. \( (1P) \)

(c) Apparently 0 and 1 are fixed points of the two-fold nested map

\[ x_{n+2} = f(f(x_n)) = 2x^2 - x^4 \]

which has four fixed points \( (1P) \)

\[ x_0^* = 0, \quad x_1^* = 1, \quad x_{2,3}^* = -\frac{1}{2} \left( 1 \pm \sqrt{5} \right). \]

Checking again \( |f'(f(x))| - |4x - 4x^3| \) at these fixed points shows that the first two are stable while \( x_{2,3}^* \) is unstable. Therefore, the sequence 0, 1, 0, 0, 1, ... is an attractive (period-doubled) fixed point. \( (1P) \)

Solutions Sheet 1
Note: Increasing $a$ further you will find that at $a \approx 1.25$ the fixed points $x_{0,1}^*$ become unstable and that the system bifurcates into period-quadrupling fixed points. This process of period-doubling continues until the system becomes chaotic.

$(\Sigma = 6P)$