Sample Solutions Exercise 1

Warmup Exercise

Exercise 1.1: Attractor of a quadratic map (6P)

Let us consider the nonlinear map $x_{n+1} = f(x_n)$ with $f(x) = a - x^2$, where $a > 0$ is a constant.

(a) Determine the real-valued fixed points $x^* = f(x^*)$ and assess their stability. (2P)

(b) Set $a := 1$ and $x_0 := 0.5$ and iterate the map numerically up to $n = 20$. What happens for $n \to \infty$ and how do you interpret the result? (2P)

(c) Prove your observation in (b) analytically. (2P)

Sample Solution

(a) Solving $x^* = a - (x^*)^2$ we find (for $a > 0$) two real-valued the fixed points (1P)

$$x^*_\pm = \frac{1}{2}\left(\pm\sqrt{1+4a} - 1\right).$$

In order to check the stability we compute (1P)

$$f'(x^*_\pm) = 2x^*_\pm = \pm\sqrt{1+4a} - 1$$

A fixed point is stable if $|f'(x)| < 1$. Therefore x^*_+ is stable for $a < \frac{3}{4}$, marginal at $a = \frac{3}{4}$, and unstable for $a > \frac{3}{4}$. The other fixed point x^*_- is always unstable. (1P)

(b) For $a = 1$ and $x_0 = 0.5$ the iteration sequence reads (can be computed even with a pocket calculator): 0.5 | 0.75 | 0.4375 | 0.808594 | 0.346176 | 0.880162 | 0.225315 | 0.949233 | 0.0989562 | 0.990208 | 0.0194888 | 0.99962 | 0.00075948 | 0.999999 | $1.15362 * 10^{-6}$ | 1. | $2.66165 * 10^{-12}$ | 1. | 0. | 1. | 0. | ... (1P)

That is, until sufficiently many iterations, the sequence effectively alternates between 0 and 1. This phenomenon is called *period doubling*. (1P)

(c) Apparently 0 and 1 are fixed points of the two-fold nested map

$$x_{n+2} = f(f(x_n)) = 2x^2 - x^4$$

which has four fixed points (1P)

$$x^*_0 = 0\,, \quad x^*_1 = 1\,, \quad x^*_{2,3} = -\frac{1}{2}\left(1 \pm \sqrt{5}\right).$$

Checking again $|[f(f(x))]'| - |4x - 4x^3|$ at these fixed points shows that the first two are stable while $x^*_{2,3}$ is unstable. Therefore, the sequence $0, 1, 0, 0, 1, \ldots$ is an attractive (period-doubled) fixed point. (1P)

Note: Increasing a further you will find that at $a \approx 1.25$ the fixed points $x^*_{0,1}$ become unstable and that the system bifurcates into period-quadrupling fixed points. This process of period-doubling continues until the system becomes chaotic.

($\Sigma = 6$P)