Exercise 7.1: Previous knowledge in the form of constraints (6P)

Consider a random variable with three possible values $X \in \{1, 2, 3\}$. Let $P(1), P(2), P(3)$ be the corresponding discrete probability distribution. The purpose of this exercise is to demonstrate that in the presence of constraints (Zwangsbedingungen), the probability distribution is the one which maximizes the Shannon entropy $H$ under these constraints.

(a) Use the method of Lagrange multipliers\(^1\) to find the values of $P(1), P(2), P(3)$ for which the Shannon entropy $H = -\sum_{n=1}^{3} P(n) \ln P(n)$ becomes extremal under the constraint that the probability distribution is normalized: $\sum_{n=1}^{3} P(n) = 1$. Compute the expectation value $\langle x \rangle$ and the variance $\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2$ for this solution. (2P)

(b) Repeat the calculation with the additional constraint that the expectation value $\langle x \rangle = \sum_{n=1}^{3} n P(n)$ equals 4/3. Compute the variance $\sigma^2$ for this solution. Note that you need two Lagrange multipliers in this case. (2P)

(c) Determine the normalized probability distribution under the constraints $\langle x \rangle = 4/3$ and a fixed given variance $\sigma^2$. Determine the range in which $\sigma^2$ can be chosen so that the solution is a valid probability distribution. (2P)

You may use Mathematica\(^\text{®} \) or similar algebraic computer systems to solve the systems of equations in this exercise.

Exercise 7.2: Random switch (6P)

Consider a switch which is either on (1) or off (0). Assume that the switch is initially off at $t = 0$ and changes its state randomly with the rates $w_{0 \to 1}$ and $w_{1 \to 0}$.

(a) Write down the master equation and compute $P_0(t)$ and $P_1(t)$. (2P)

(b) Compute the Shannon entropy $H(t)$ for symmetric rates $w_{0 \to 1} = w_{1 \to 0} = 1$. Check the Second Law by plotting $H(t)$ in the range $t = 0...2$. (2P)

(c) For arbitrary rates the Second Law does not apply. Find a condition for which the entropy first increases, then reaches a maximum at a finite time $t_{\max}$ which is followed by a decrease. Compute $t_{\max}$ and $H(t_{\max})$ as well as the asymptotic saturation value $\lim_{t \to \infty} H(t)$. (2P)

($\Sigma = 12$P)

To be handed in on Monday, December 02, at the beginning of the lecture (PG1 directly to Pascal).

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\(^1\)see e.g. https://de.wikipedia.org/wiki/Lagrange-Multiplikator