
Constraints [Wikimedia]

## Exercise 7.1: Previous knowledge in the form of constraints (6P)

Consider a random variable with three possible values $X \in \{1, 2, 3\}$. Let $P(1), P(2), P(3)$ be the corresponding discrete probability distribution. The purpose of this exercise is to demonstrate that in the presence of constraints (*Zwangsbedingungen*), the probability distribution is the one which maximizes the Shannon entropy $H$ under these constraints.

(a) Use the method of Lagrange multipliers[1] to find the values of $P(1), P(2), P(3)$ for which the Shannon entropy $H = -\sum_{n=1}^{3} P(n) \ln P(n)$ becomes extremal under the constraint that the probability distribution is normalized: $\sum_{n=1}^{3} P(n) = 1$. Compute the expectation value $\langle x \rangle$ and the variance $\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2$ for this solution. (2P)

(b) Repeat the calculation with the *additional* constraint that the expectation value $\langle x \rangle = \sum_{n=1}^{3} nP(n)$ equals 4/3. Compute the variance $\sigma^2$ for this solution. Note that you need two Lagrange multipliers in this case. (2P)

(c) Determine the normalized probability distribution under the constraints $\langle x \rangle = 4/3$ and a fixed given variance $\sigma^2$. Determine the range in which $\sigma^2$ can be chosen so that the solution is a valid probability distribution. (2P)

*You may use Mathematica® or similar algebraic computer systems to solve the systems of equations in this exercise.*

## Exercise 7.2: Random switch (6P)


Consider a switch which is either on (1) or off (0). Assume that the switch is initially off at $t = 0$ and changes its state randomly with the rates $w_{0\to 1}$ and $w_{1\to 0}$.

(a) Write down the master equation and compute $P_0(t)$ and $P_1(t)$. (2P)

(b) Compute the Shannon entropy $H(t)$ for symmetric rates $w_{0\to 1} = w_{1\to 0} = 1$. Check the Second Law by plotting $H(t)$ in the range $t = 0...2$. (2P)

(c) For arbitrary rates the Second Law does not apply. Find a condition for which the entropy first increases, then reaches a maximum at a finite time $t_{max}$ which is followed by a decrease. Compute $t_{max}$ and $H(t_{max})$ as well as the asymptotic saturation value $\lim_{t\to\infty} H(t)$. (2P)

**($\Sigma$ = 12P)**

To be handed in on Monday, December 02, at the beginning of the lecture (PG1 directly to Pascal).

[1] see e.g. https://de.wikipedia.org/wiki/Lagrange-Multiplikator