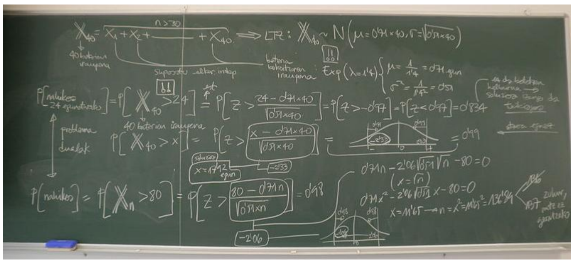


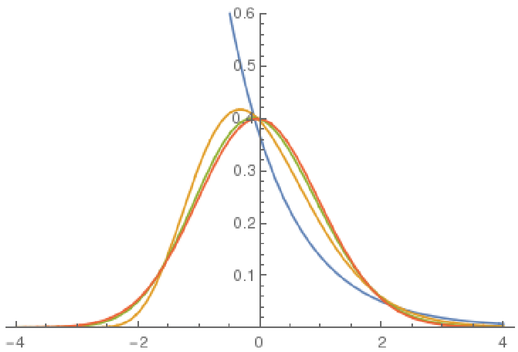
Central limit theorem [Joxemai, Wikimedia]

Exercise 4.1: Central Limit Theorem (CLT) (6P)

Let $X_1, X_2, \ldots, X_N$ be statistically independent and identically distributed random variables with the probability density function

$$p_X(x) = \begin{cases} e^{-x} & \text{if } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

With this exercise we would like to demonstrate the CLT for $Z_N = \sum_{i=1}^N X_i$ in the limit $N \to \infty$.



(a) Show that the N-fold convolution product of $p(x)$ is given by (1P)

$$p_{Z_N}(z) \;=\; p_X^{*N}(z) \;=\; \big(\underbrace{p_X * p_X * \cdots * p_X}_{N\,\text{times}}\big)(z) \;=\; \frac{z^{N-1}}{(N-1)!}\, e^{-z} \qquad (z \geq 0)\,.$$

(b) Check the norm and compute the mean and the variance of $p_{Z_N}(z)$. (1P)

(c) Standardize the distribution by shifting and rescaling Z_N in such a way that the new random variable $\tilde{Z}_N$ has a probability density $p_{\tilde{Z}_N}(z)$ with unit norm, zero mean, and unit variance (see figure). (1P)

(d) Show that the cumulant-generating function $K_{\tilde{Z}}(t)$ of the standardized probability density $p_{\tilde{Z}_N}(z)$ is given by (1P)

$$K_{\tilde{Z}}(t) \;=\; \frac{1}{2} N \ln N - t\sqrt{N} - N \ln\left(\sqrt{N} - t\right)\,.$$

(e) Compute all cumulants κ_n and show that only κ_2 survives in the limit $N \to \infty$. What does it mean? (2P)

⇒ **Please turn over**

Exercise 4.2: Reconstruction of a probability density (4P)

(a) Prove the following statement: If the moment-generating function $M_X(t)$ is analytic, then the corresponding probability density $p(x)$ is given by the inverse Fourier transform (1P)

$$p(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}s \, e^{-ixs} M_X(is).$$

(b) Consider a probability distribution with the cumulants

$$\{\kappa_0, \kappa_1, \kappa_2, \kappa_3, \ldots\} = \{0, 0, \frac{3}{2}, 0, -\frac{1}{2}, 0, \frac{1}{3}, 0, -\frac{1}{4}, 0, \frac{1}{5}, 0, -\frac{1}{6}, \ldots\}$$

Compute the generating functions $K(t)$ and $M(t)$. (2P)

(c) Use (a) to reconstruct the probability density $p(x)$. (1P)

Exercise 4.3: Transformation of probability densities (2P)

Let X and Y be two uncorrelated random variables which are both distributed according to a normal distribution with zero mean and unit variance. What is the probability density of the random variable $Z := X/Y$?

(Σ = 12P)

To be handed in on Monday, November 11, at the beginning of the lecture (PG1 directly to Pascal).