Statistical Physics & Thermodynamics

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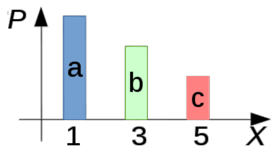
Exercise 3.1: Reconstruct probabilities from given moments (4P)

Consider a system with three configurations $\Omega = \{a, b, c\}$ together with an associated map $X : \Omega \to \mathbb{R}$ defined by



$$X_a = 1, \quad X_b = 3, \quad X_c = 5\,.$$

(a) How many moments do you have to know in order to be able to reconstruct the probabilitiy distribution $\{P_a, P_b, P_c\}$ and why?

(b) Suppose that M_1 and M_2 are given (with the map X defined above). Find the solution for the probabilities P_a, P_b, and P_c.

(c) Not all possible values for M_1 are M_2 lead to consistent results. Give a counter-example.

(d) Let $M_1 = \frac{7}{2}$ and $M_2 = 15$. Compute the probabilities P_a, P_b, P_c and find a general formula for the higher moments M_n with $n = 3, 4, \ldots \infty$.

Exercise 3.2: Poisson distribution (6P)

The poisson distribution $P_\lambda(k) = \frac{\lambda^k}{k!} e^{-\lambda}$ can be understood as the limit of the binomial distribution in the case of "rare events".

(a) Let $p = \lambda/N$ and take $N \to \infty$ while keeping λ and k constant. Show that in this limit we can approximate $(1-p)^{N-k} \approx e^{-\lambda}$. (1P)

(b) Show similarly that $\binom{N}{k} \approx \frac{N^k}{k!}$. (1P)

(c) Use (a) and (b) to show that in this limit the binomial distribution tends to the Poisson distribution. (1P)

(d) Check that the Poisson distribution is properly normalized. (1P)

(e) Compute the moment- and cumulant-generating functions. (1P)

(f) Determine all cumulants. (1P)

Exercise 3.3: Transformation of a probability density (2P)

Suppose that the random variable $X \in [0, 2]$ is distributed according to the probability density $p_X(x) = \frac{3}{4}x(2-x)$. How is the random variable $Y = X^2$ distributed? Check the normalization of $p_Y(y)$.

(Σ = 12P)

To be handed in on Monday, November 04, at the beginning of the lecture (PG1 directly to Pascal).