EXERCISE 2.1: BAYES THEOREM (2P)

An urn contains three red and four black balls. Two balls are randomly drawn from the urn (without putting them back). What is the probability that the first one is black given that the second one is red?

SAMPLE SOLUTION

Let A be the event that the first drawn ball is black and B that the second drawn ball is red. As we have initially 4 black and 3 red balls in the urn, the probability for the first ball to be black is $P(A) = 4/7$. Likewise the probability that the second ball is red (not knowing the first one) is $P(B) = 3/7$. The conditional probability $P(B|A)$ is the probability that the second one is red, given that the first one was black. This means that there are still three black and three red balls in the urn, hence $P(B|A) = 1/2$. The 'reversed' conditional probability $P(A|B)$ can be computed using Bayes theorem

$$P(A|B) = \frac{P(A)}{P(B)} P(B|A) = \frac{2}{3}.$$

EXERCISE 2.2: RECONSTRUCT PROBABILITIES FROM GIVEN MOMENTS (4P)

Consider a system with three configurations $\Omega = \{a, b, c\}$
together with an associated map $X : \Omega \to \mathbb{R}$ defined by

$$X_a = 1, \quad X_b = 3, \quad X_c = 5\,.$$

(a) How many moments do you have to know in order to be able to reconstruct the probabilitiy distribution $\{P_a, P_b, P_c\}$ and why?

(b) Suppose that M_1 and M_2 are given (with the map X defined above). Find the solution for the probabilities P_a, P_b, and P_c.

(c) Not all possible values for M_1 are M_2 lead to consistent results. Give a counter-example.

(d) Let $M_1 = \frac{7}{2}$ and $M_2 = 15$. Compute the probabilities P_a, P_b, P_c and find a general formula for the higher moments M_n with $n = 3, 4, \ldots \infty$.

SAMPLE SOLUTION

(a) Two! We have three unknowns $\{P_a, P_b, P_c\}$ and therefore we need three equations in total. One of them is the normalization condition $P_a + P_b + P_c = 1$. Therefore, we have to know two moments which provide remaining two equations.

(b) Solving the equations[1]

$$\begin{aligned} 1 &= P_a + P_b + P_c \\ M_1 &= P_a X_a + P_b X_b + P_c X_c \\ M_2 &= P_a X_a^2 + P_b X_b^2 + P_c X_c^2 \end{aligned}$$

we find the solution

$$\begin{aligned} P_a &= \frac{M_2 + X_b X_c - M_1(X_b + X_c)}{(X_a - X_b)(X_b - X_c)} &= \frac{1}{8}(15 - 8M_1 + M_2) \\ P_b &= \frac{M_2 + X_a X_c - M_1(X_a + X_c)}{(X_a - X_b)(X_c - X_b)} &= \frac{1}{4}(-5 + 6M_1 - M_2) \\ P_c &= \frac{M_2 + X_a X_b - M_1(X_a + X_b)}{(X_a - X_c)(X_b - X_c)} &= \frac{1}{8}(3 - 4M_1 + M_2) \end{aligned}$$

(c) Probabilites must not be negative or larger than 1. A possible conterexample is e.g. $M_1 = 1, M_2 = 2$ where $P_b = -1/4$. There are many more examples of that kind.

(d) Inserting the solution into $M_n = P_a X_a^n + P_b X_b^n + P_c X_c^n$ we get

$$M_n = \frac{1}{4}(6M_1 - M_2 - 5)3^n + \frac{1}{8}(-4M_1 + M_2 + 3)5^n + \frac{1}{8}(-8M_1 + M_2 + 15)$$

For $n = 1, 2$ this expression reduces correctly to M_1, M_2.

Note: The purpose is to make the students aware of the fact that knowing all moments is like knowing the whole probability distribution (up to subtle mathematical exceptions).

Exercise 2.3: The binomial distribution (6P)

The binomial distribution reads

$$P_{N,p}(k) = \binom{N}{k} p^k (1-p)^{N-k}\,.$$

(a) Use the relation $\binom{N}{k} = \frac{N}{k}\binom{N-1}{k-1}$ to derive a recursion relation for the (non-centralized) moments $m_n(N,p) = \langle k^n \rangle_{N,p}$ of the binomial distribution. This recursion relation should have the following form: (2P)

$$m_n(N,p) = \text{ some function of } \Big(m_0(N-1,p), m_1(N-1,p), \ldots, m_{n-1}(N-1,p)\Big)$$

(b) Apply this recursion relation to compute the first four non-centralized moments $m_0, \ldots, m_3$ explicitly. (2P)

(c) Compute the moment-generating function $M_{N,p}(t) = \langle e^{kt} \rangle_{N,p}$ of the binomial distribution analytically (please provide a complete proof). (1P)

[1] We encourage the use of algebraic software packages like *Mathematica*®. But please do not submit uncommented notebooks as solutions.

(d) Compute the first three cumulants $\kappa_0, \ldots, \kappa_3$ explicitly from the cumulant-generating function $K_{N,p}(t) = \ln M_{N,p}(t)$. You are encouraged to use *Mathematica*®or similar computer algebra software. (1P)

Sample Solution

(a) Start with the definition:

$$\begin{aligned} m_n(N,p) = \langle k^n \rangle_{N,p} &= \sum_{k=0}^{N} k^n \binom{N}{k} p^k (1-p)^{N-k} = \sum_{k=1}^{N} k^n \frac{N}{k} \binom{N-1}{k-1} p^k (1-p)^{N-k} \\ &= N \sum_{k=0}^{N-1} (k+1)^{n-1} \binom{N-1}{k} p^{k+1} (1-p)^{N-k-1} \end{aligned}$$

Now we expand $(k+1)^{n-1} = \sum_{q=0}^{n-1} \binom{n-1}{q} k^q$ to obtain

$$\begin{aligned} m_n(N,p) &= Np \sum_{q=0}^{n-1} \binom{n-1}{q} \sum_{k=0}^{N-1} k^q \binom{N-1}{k} p^k (1-p)^{N-1-k} \\ &= Np \sum_{q=0}^{n-1} \binom{n-1}{q} m_q(N-1,p) \end{aligned}$$

(b) The zeroth moment is the norm which anchors the recursion:

$$m_0(N,p) = \sum_{k=0}^{N} \binom{N}{k} p^k (1-p)^{N-k} = (p+1-p)^N = 1$$

The first moment is the mean:

$$m_1(N,p) = Npm_0(N,p) = Np$$

The second and the third moment can be computed similarly, giving:

$$\begin{aligned} m_2(N,p) &= Np\,(Np - p + 1) \\ m_3(N,p) &= Np\left(N^2p^2 - 3Np^2 + 3Np + 2p^2 - 3p + 1\right). \end{aligned}$$

(c)

$$\begin{aligned} M(t;N,p) = \langle e^{kt} \rangle_{N,p} &= \sum_{k=0}^{N} e^{kt} \binom{N}{k} p^k (1-p)^{N-k} \\ &= \sum_{k=0}^{N} \binom{N}{k} (p\,e^t)^k (1-p)^{N-k} = (pe^t + (1-p))^N. \end{aligned}$$

(d) The Mathematica command

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Series[Log[M[t]], {t, 0, 3}]
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gives the output:

$$Npt + \frac{1}{2}t^2\left(Np - Np^2\right) + \frac{1}{6}t^3\left(2Np^3 - 3Np^2 + Np\right) + O\left(t^4\right)$$

From this we can read off the cumulants

$$\kappa_0 = 0, \qquad \kappa_1 = Np, \qquad \kappa_2 = Np(1-p), \qquad \kappa_3 = Np(1-p)(1-2p)$$

(Σ = 12P)