Statistical Physics & Thermodynamics

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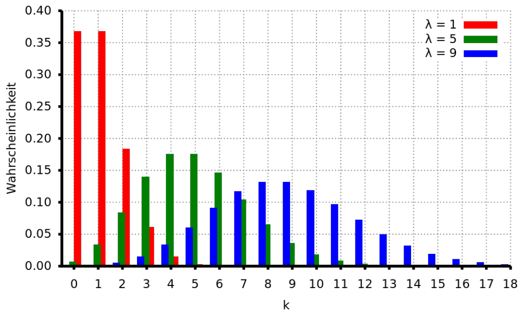


Poisson distribution for $\lambda = 1, 5$, and 9.

Exercise 4.1: Poisson distribution (6P)

The poisson distribution $P_\lambda(k) = \frac{\lambda^k}{k!} e^{-\lambda}$ can be understood as the limit of the binomial distribution in the case of "rare events".

(a) Let $p = \lambda/N$ and take $N \to \infty$ while keeping λ and k constant. Show that in this limit we can approximate $(1-p)^{N-k} \approx e^{-\lambda}$. (1P)

(b) Show similarly that $\binom{N}{k} \approx \frac{N^k}{k!}$. (1P)

(c) Use (a) and (b) to show that in this limit the binomial distribution tends to the Poisson distribution. (1P)

(d) Check that the Poisson distribution is properly normalized. (1P)

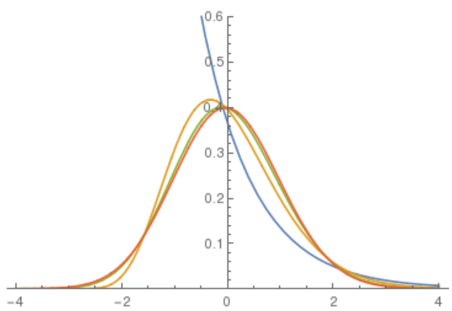
(e) Compute the moment- and cumulant-generating functions. (1P)

(f) Determine all cumulants. (1P)

Exercise 4.2: Central Limit Theorem (CLT) (6P)

Let $X_1, X_2, \ldots, X_N$ be statistically independent and identically distributed random variables with the probability density function

$$p_X(x) = \begin{cases} e^{-x} & \text{if } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$



With this exercise we would like to demonstrate the CLT for $Z_N = \sum_{i=1}^N X_i$ in the limit $N \to \infty$.

(a) Show that the N-fold convolution product of $p(x)$ is given by (1P)

$$p_{Z_N}(z) \;=\; p_X^{*N}(z) \;=\; \big(\underbrace{p_X * p_X * \cdots * p_X}_{N \text{ times}}\big)(z) \;=\; \frac{z^{N-1}}{(N-1)!}\, e^{-z} \qquad (z \geq 0)\,.$$

(b) Check the norm and compute the mean and the variance of $p_{Z_N}(z)$. (1P)

(c) Standardize the distribution by shifting and rescaling Z_N in such a way that the new random variable $\tilde{Z}_N$ has a probability density $p_{\tilde{Z}_N}(z)$ with unit norm, zero mean, and unit variance (see figure). (1P)

(d) Show that the cumulant-generating function $K_{\tilde{Z}}(t)$ of the standardized probability density $p_{\tilde{Z}_N}(z)$ is given by (1P)

$$K_{\tilde{Z}}(t) = \frac{1}{2} N \ln N - t\sqrt{N} - N \ln\left(\sqrt{N} - t\right) .$$

(e) Compute all cumulants κ_n and show that only κ_2 survives in the limit $N \to \infty$. What does it mean? (2P)

(Σ = 12P)

To be handed in on Monday, November 13, at the beginning of the lecture.