SAMPLE SOLUTIONS EXERCISE 11

EXERCISE 11.1: STATISTICS OF MISPRINTS (3P)

In a printing company, the error probability for a typo is $q = 10^{-6}$. The typos occur completely uncorrelated and the probability is the same for all characters.

(a) Estimate the numerical probability to have less than 5 typos in a book with 10^7 characters. (1P)

(b) Let n be the number of correct characters between two subsequent typos. Compute the probability distribution $P(n)$ in an infinitely long text. (1P)

(c) Determine the mean value and the variance of this distribution. (1P)

SAMPLE SOLUTION

(a) The number of typos is distributed according to a binomial distribution. However, since the number of characters is large and the error rate is low, and since the total number of the characters in the book is not specified, we have to use the Poisson distribution (the Poisson distribution is obtained from the binomial distribution in the limit of rare events). More specifically, if N is the total number of characters in the book, the probability of having n typos can be estimated as

$$P(n) \approx \frac{\lambda^n}{n!} e^{-\lambda},$$

where $\lambda = Nq$. The probability to have less than 5 typos is

$$P(0) + P(1) + P(2) + P(3) + P(4) \approx 0.292527.$$

(b) Suppose we just had a typo. Then the probability to get n correct characters followed by another typo is

$$P(n) = (1-q)^n q.$$

As can be verified easily, this probability distribution is correctly normalized.

(c) To solve this exercise one needs the geometric series and its moments:

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}, \quad \sum_{n=0}^{\infty} n\, a^n = \frac{a}{(1-a)^2}, \quad \sum_{n=0}^{\infty} n^2 a^n = \frac{a(1+a)}{(1-a)^3}.$$

This gives:

$$\langle n \rangle = \sum_{n=0}^{\infty} nP(n) = \frac{1-q}{q}$$

$$VAR(n) = \langle n^2 \rangle - \langle n \rangle^2 = \sum_{n=0}^{\infty} n^2 P(n) - \frac{(1-q)^2}{q^2} = \frac{1-q}{q^2}.$$

The numerical values are:

$$\langle n \rangle = 999\,999\,, \quad VAR(n) = 999\,999\,000\,000.$$

Thus, the mean value is roughly q^{-1} and the width (square root of variance) is of the same order.

Exercise 11.2: Ultrarelativistic gas (9P)

The energy of a relativistic particle is given by $E = \sqrt{m^2c^4 + p^2c^2}$, where c is the velocity of light and m is the rest mass. A gas is called *ultrarelativistic* if the energy of the particles is so high that the rest mass can be neglected. In the following let us consider an ultrarelativistic gas with N particles in a container with the volume V:

(a) Show that the phase space volume $\Phi = \int_{E<E_0} \prod_{i=1}^{N} \mathrm{d}^3q_i\, \mathrm{d}^3p_i$ over all states with the total energy $E < E_0$ can be written as

$$\Phi = (4\pi V)^N I_N(E_0)\,,$$

where I_N obeys the recursion relation (3P)

$$I_N(E) = \int_0^{E/c} \mathrm{d}p\, p^2\, I_{N-1}(E - pc).$$

(b) Verify that $I_N(E)$ is a homogeneous function and show that it can be written as $I_N(E) = (E/c)^{3N} C_N$, where C_N is a (N-dependent) constant. (1P)

(c) Insert the homogeneity obtained in (b) into the recursion relation given in (a) and solve the integral (e.g. with *Mathematica*®). This gives a recursion relation for the C_N which you can solve exactly. (1P)

(d) Use this result to determine the number of states in an infinitesimal energy shell $[E, E + \delta E]$. (1P)

(e) Using Stirling's formula and taking the permutation entropy of non-distinguishable quantum particles into account, show that (1P)

$$H(E, V, N) \simeq N\left[\ln\left(\frac{8\pi V E^3}{27 h^3 c^3 N^4}\right) + 4\right] + \ldots.$$

(f) Compute temperature and pressure and show that the equation of state is the same as the one of the non-relativistic ideal gas. (1P)

(g) Use (f) to show that the entropy can be expressed as

$$H(E, V, N) = N \ln\left(\frac{AT^3V}{N}\right), \qquad H(T, P, N) = N \ln\left(\frac{AT^4}{P}\right)$$

where A is a constant. Use these formulas to calculate the specific heats C_V and C_P. How does $\gamma = C_P/C_V$ differ from the non-relativistic case? (1P)

Sample Solution

(a) First, the integration over the volume can be carried out, giving

$$\Phi(E<E_0) \;=\; \int_{E<E_0} \prod_{i=1}^{N} \mathrm{d}^3q_i\,\mathrm{d}^3p_i \;=\; V^N \int_{E<E_0} \prod_{i=1}^{N} \mathrm{d}^3p_i$$

Since the gas is ultrarelativistic, we can approximate the energy by $E = c\sum_{i=1}^{N} p_i$, where $p_i = |\vec{p}_i|$. Therefore, we can rewrite the above expression in terms of $3N$ unconstrained integrals times a δ that filters the contribution with $E < E_0$:

$$\Phi(E<E_0) \;=\; V^N \int_{-\infty}^{+\infty} d^3p_1 \cdots \int_{-\infty}^{+\infty} d^3p_N\, \delta\big[c\sum_{i=1}^{N} p_i < E_0\big]$$

These integrals can be evaluated by rewriting the expression in 3D spherical coordinates $\mathrm{d}^3p_0 \to 4\pi p_i^2\,\mathrm{d}p_i$:

$$\Phi(E<E_0) \;=\; (4\pi V)^N \underbrace{\int_0^{\infty} p_1^2\,\mathrm{d}p_1 \cdots \int_0^{\infty} p_N^2\,\mathrm{d}p_N\, \delta\big[c\sum_{i=1}^{N} p_i < E_0\big]}_{I_N(E_0)}$$

Pulling the last integral in front this turns into

$$\Phi(E<E_0) \;=\; (4\pi V)^N \int_0^{E_0/c} p_N^2\,\mathrm{d}p_N \underbrace{\int_0^{\infty} p_1^2\,\mathrm{d}p_1 \cdots \int_0^{\infty} p_{N-1}^2\,\mathrm{d}p_{N-1}\;\delta\big[c\sum_{i=1}^{N-1} p_i < E_0 - cp_N\big]}_{I_{N-1}(E_0 - cp_n)}$$

Comparing the two expressions we arrive at the wanted recursion relation

$$I_N(E) = \int_0^{E/c} \mathrm{d}p\, p^2\, I_{N-1}(E-pc).$$

(b) Let $\lambda > 0$ be a dilation factor:

$$\begin{aligned}
I_N(\lambda E) &= \int_0^{\infty} p_1^2\,\mathrm{d}p_1 \cdots \int_0^{\infty} p_N^2\,\mathrm{d}p_N\, \delta\big[c\sum_{i=1}^{N} p_i < \lambda E\big] \\
&= \int_0^{\infty} p_1^2\,\mathrm{d}p_1 \cdots \int_0^{\infty} p_N^2\,\mathrm{d}p_N\, \delta\big[c\sum_{i=1}^{N} p_i/\lambda < E\big] \quad \Big|p_i \to \lambda p_i \\
&= \lambda^{3N} \int_0^{\infty} p_1^2\,\mathrm{d}p_1 \cdots \int_0^{\infty} p_N^2\,\mathrm{d}p_N\, \delta\big[c\sum_{i=1}^{N} p_i < \lambda E\big] = \lambda^{3N} I_N(E).
\end{aligned}$$

By setting $\lambda = 1/E$ the homogeneity implies that $I_N(E)$ is essentially given by a power-law

$$I_N(E) = E^{3N} I_N(1).$$

Setting $C_N = c^{3N} I_N(1)$ we arrive at the relation $I_N(E) = C_N (E/c)^{3N}$, where C_N is a constant.

(c) We insert the relation $C_N = c^{3N} I_N(1)$ into the recursion relation given in (a):

$$C_N = \Big(\frac{c}{E}\Big)^{3N} \int_0^{E/c} \mathrm{d}p\, p^2\, C_{N-1}\Big(\frac{E}{c} - p\Big)^{3N-3}$$
$$= \int_0^1 \mathrm{d}x\, x^2\, (1-x)^{3N-3}\, C_{N-1} = \frac{2(3N-3)!}{(3N)!} C_{N-1}$$

As can be seen, this recursion is anchored at $C_1 = \int_0^1 x^2\, \mathrm{d}x = \frac{1}{3}$ in it is obvious that the recursion relation is solved by the exact solution

$$C_N = \frac{2^N}{(3N)!}.$$

(d) Inserting this result back into the formula given in (a), the phase space volume reads

$$\Phi(E) = (4\pi V)^N\, I_N(E) = \frac{1}{(3N)!}\Big(\frac{8\pi V E^3}{c^3}\Big)^N.$$

The phase space volume $\delta\Phi$ of the energy shell in the range $[E, E+\delta E]$ is then given by

$$\delta\Phi(E, \delta E) = \frac{\partial \Phi(E)}{\partial E}\delta E = \frac{1}{(3N-1)!}\Big(\frac{8\pi V E^3}{c^3}\Big)^N \frac{\delta E}{E}.$$

The number of states within this energy shell is simply given by $\delta\Phi(E, \delta E)/h^{3N}$, where $h = \hbar/(2\pi)$ is the Planck constant.

(e)

$$H(E, V, N) = k \ln\Big(\frac{\delta\Phi}{h^{3N} N!}\Big) \simeq N \left[\ln\Big(\frac{8\pi V E^3}{27 h^3 c^3 N^4}\Big) + 4\right] + \mathcal{O}(\text{subordinate terms})$$

(f) We can compute both quantities by taking the derivatives

$$\beta = \frac{1}{T} = \left(\frac{\partial H}{\partial E}\right)_{V,N} = \frac{3N}{E}\,, \qquad P = T\left(\frac{\partial H}{\partial V}\right)_{E,N} = \frac{NT}{V}.$$

The second equation means that $PV = NT$ (or $pV = N k_B T$ in textbook units), i.e., we get the same equation of state as in the non-relativistic ideal gas. In addition, we get $E = 3NT$.

(g) Inserting $E = 3NT$ and $V = NP/T$ one arrives with $A = \frac{8\pi}{h^3 c^3}$ at the given formulas. Calculating the derivative the factors inside the logarithm are like constants. The result reads:

$$C_V = T\left(\frac{\partial H}{\partial T}\right)_{V,N} = 3N\,, \qquad C_P = T\left(\frac{\partial H}{\partial T}\right)_{P,N} = 4N.$$

Thus the ratio is given by $\gamma = C_P/C_V = 4/3$, which has to be compared with $\gamma = 5/3$ in the non-relativistic case.

(Σ = 12P)