Statistical Physics & Thermodynamics

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Exercise 5.1: Playing with black holes (4P)

A (non-rotating electrically neutral) black hole is space-time singularity described by a single parameter, namely, its mass M. It is surrounded by a spherical event horizon with radius $r = 2GM/c^2$ and surface area $A = 4\pi r^2$, where G is the gravitational constant.[1]

(a) Suppose that a black hole has a certain entropy H depending on M. Compute the black hole temperature T for given $H(M)$ using the Clausius relation $\mathrm{d}E = T\,\mathrm{d}H$ and Einsteins $E = Mc^2$. (1P)

(b) A black hole with temperature $T > 0$ is not black: it is expected to emit radiation like a black body. According to Wien's displacement law we expect a typical wave length $\lambda \approx \hbar c/4k_B T$. Since the black hole has only a single relevant length scale, namely, its horizon radius r, it is natural to conjecture that $\lambda \approx r$. Use this conjecture to compute $T(M)$ and $H(M)$. (1P)

(c) Show that $H(M)$ is proportional to the surface area of the black hole. (1P)

(d) Ordinary physical units (m,s,kg) are superfluous since ther are natural Planck units. For example, the Planck length and area are given by $l_P = \sqrt{\hbar G/c^3}$ and $A_P = l_P^2$. How many bits per unit Planck area reside on the surface of a black hole? (1P)

Sample Solution

The following solution is written using the physics convention $H = k_B \ln|\Omega|$. Other conventions such as the information-theoretic $H = \log_2(|\Omega|)$ differ only by a constant factor.

(a) If we combine

$$E = Mc^2 \quad \Rightarrow \quad \mathrm{d}E = c^2\,\mathrm{d}M$$

with the Clausius relation $\mathrm{d}E = T\,\mathrm{d}H$ we get the relation

$$c^2\,\mathrm{d}M = T\,\mathrm{d}H \quad \Rightarrow \quad T = c^2\left(\frac{\mathrm{d}H(M)}{\mathrm{d}M}\right)^{-1} = \frac{c^2}{H'(M)}$$

(b) Setting $\lambda = r = 2GM/c^2$ we get

$$T(M) = \frac{\hbar c}{4k_B\lambda} = \frac{\hbar c^3}{8k_B GM}.$$

Then we can compute $H'(M)$ and integrate it:

$$H'(M) = \frac{c^2}{T(M)} = \frac{8k_B GM}{\hbar c} \quad \Rightarrow \quad H(M) = \frac{4k_B GM^2}{\hbar c}$$

[1] This exercise uses simplified assumptions. The results will differ from the correct results obtained by Hawking and Bekenstein by constant factors such as π.

(c) The black hole has the surface area

$$A = 4\pi r^2 = \frac{16\pi G^2 M^2}{c^4} \quad \Rightarrow \quad M^2 = \frac{c^4 A}{16\pi G^2}$$

Inserting this back into the result of (b) we can express the entropy H as

$$H = \frac{k_B c^3}{4\pi \hbar G} \propto A.$$

This is Hawking's main result: The entropy of a black hole is proportional to its horizon area.

(d) Since $A_P = \hbar G/c^3$ we can rewrite

$$H = \frac{k_B}{4\pi} \frac{A}{A_P}.$$

Since there are $N = A/a_P$ Planck areas on the surface, the entropy $h = H/N$ per Planck area is

$$h = H/N = \frac{k_B}{4\pi}.$$

This is the entropy in physics units (using k_B and ln). To get the number of bits we simply have to divide by $k_B \ln 2$:

$$h[bits] = \frac{1}{4\pi \ln 2}$$

A full calculation gives the same formulat without π. The main point is that one gets 'of the order of 1' bits per planck area.

Exercise 5.2: Waiting for an unlikely event **(4P)**

Consider a random experiment which yiels the result 'A' with probability $p < 1$ and 'B' otherwise. Let X be a random variable defined as the number of trials *before* you get the first 'B' (for example, 'B'$\mapsto 0$, 'AB'$\mapsto 1$, 'AAB' $\mapsto 2, \ldots$).

(a) Specifiy the probability distribution $P_X : \mathbb{N}_0 \to [0,1] : \; x \mapsto P_X(x)$. (1P)

(b) Check the normalization of your result in (a). (1P)

(c) Determine the average entropy $H_X = \langle H_X(x) \rangle_x$. How many bits of information do you get on average for $p = 0.5$ and $p = 0.1$? What happens in the limits $p \to 0$ and $p \to \infty$? (2P)

Sample Solution

(a) The probability to get directly '1' is $P_X(0) = 1-p$. The probability to get '01' is p times $1-p$. Likewise, the probability to get '001' is $P_X(2) = p^2(1-p)$. Thus, in general we have

$$P_X(x) = p^x(1-p).$$

(b) We sum over all $x \in \mathbb{N}_0$ and use the geometric series (which converges since $p < 1$):

$$\sum_{x=0}^{\infty} p^x(1-p) = (1-p)\sum_{x=0}^{\infty} p^x = \frac{1-p}{1-p} = 1.$$

(c) The average entropy is given by $H_X = -\sum_{x=1}^{\infty} p_X(x)\log_2 p_X(x)$ (using ln instead of $\log_2$ is also okay):

$$\begin{aligned} H_X &= -\sum_{x=0}^{\infty}(1-p)p^x \log_2\Big[(1-p)p^x\Big] \\ &= -(1-p)\sum_{x=0}^{\infty} p^x\Big(\log_2(1-p) + x\log_2(p)\Big) \\ &= -(1-p)\log_2(1-p)\underbrace{\sum_{x=0}^{\infty} p^x}_{=1/(1-p)} -(1-p)\log_2(p)\sum_{x=0}^{\infty} x p^x \end{aligned}$$

The last sum can be evaluated as follows:

$$\sum_{x=0}^{\infty} x p^x = \sum_{x=1}^{\infty} x p^x = \sum_{x=0}^{\infty}(x+1)p^{x+1} = p\sum_{x=0}^{\infty}(x+1)p^x = p\sum_{x=0}^{\infty}\frac{\mathrm{d}}{\mathrm{d}p}p^{x+1}$$

$$= p\frac{\mathrm{d}}{\mathrm{d}p}\sum_{x=0}^{\infty}p^{x+1} = p\frac{\mathrm{d}}{\mathrm{d}p}\Big(\frac{p}{1-p}\Big) = p\Big(\frac{1}{1-p} + \frac{p}{(1-p)^2}\Big) = \frac{p}{(1-p)^2}$$

Collecting all terms yields:

$$H_X = -\log_2(1-p) - \frac{p}{1-p}\log_2(p)$$

Special values are (using $\log_2$, otherwise there will be an additional factor of $\ln 2$):

p	H_X
$\to 0$	0
0.1	0.521106...
0.5	2
$\to 1$	∞

Interpretation: The larger p is, the longer the 'A'-sequence will be until the first 'B' occurs. At the same time, the fluctuations of the length of the 'A'-string will increase and finally diverge in the limit $p \to 1$.

Exercise 5.3: Uniform probability density on a unit sphere (4P)

Consider a unit sphere with standard spherical coordinates $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$. The figure shows uniformly distributed random points on the surface.

(a) Calculate the normalized probability density $p(\theta, \phi)$ which is uniform (i.e. constant per unit area) on the surface of the sphere. (1P)

(b) Show that this probability density factorizes. (1P)

(c) Let X_1 and X_2 be two uncorrelated continuous random variables with a uniform distribution between 0 and 1 (e.g. x_1, x_2 can be drawn from a standard random number generator). Transform $x_1, x_2 \in [0,1]$ to spherical coordinates θ, ϕ in such a way that the resulting probability on the unit sphere is uniform. (2P)

Sample Solution

(a) The infinitesimal surface element on a sphere reads

$$\mathrm{d}S = r^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi$$

A probability density on the unit sphere ($r = 1$) is uniform if $p(\theta,\phi)\,\mathrm{d}\theta\,\mathrm{d}\phi \propto \mathrm{d}S$, therefore

$$p(\theta,\phi) \propto \sin(\theta)\,.$$

Normalizing the probability density by

$$\int_0^\pi \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi \, p(\theta,\phi) = 1$$

we arrive at

$$p(\theta,\phi) = \frac{1}{4\pi}\sin(\theta).$$

(b) This result shows directly that $p(\theta,\phi) = p(\theta)p(\phi)$ formalally factorizes into the normalized distributions

$$p(\theta) = \frac{1}{2}\sin\theta\,, \qquad p(\phi) = const = \frac{1}{2\pi}.$$

(c) Since $p(\theta,\phi)$ factorizes, the two random variables can be transformed independently, i.e. we are now looking for two function $\theta = f(x_1)$ and $\phi = g(x_2)$ reproducing the probability density calculated above. For the polar angle ϕ this is trivial – we simply have to set

$$\phi = 2\pi z_2$$

As for the azimuthal angle θ, we start with the differential relation

$$p(\theta)\,\mathrm{d}\theta \;=\; p(z_1)\,\mathrm{d}z_1$$

and integrate it:

$$\int_0^\theta p(\theta')\,\mathrm{d}\theta' \;=\; \int_0^{z_1} p(z_1')\,\mathrm{d}z_1'\,.$$

Inserting $p(\theta') = \frac{1}{2}\sin\theta'$ and $p(z_1) = 1$ we obtain

$$\frac{1}{2}\int_0^\theta \sin\theta'\,\mathrm{d}\theta' \;=\; \int_0^{z_1} \mathrm{d}z_1' \qquad \Rightarrow \qquad \frac{1}{2}(1-\cos\theta) = z_1$$

Therefore, the final result reads

$$\boxed{\theta = \arccos(1-2z_1)\,, \quad \phi = 2\pi z_2}$$

(Σ = 12P)