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Exercise 4.1: Poisson distribution (6P)

The poisson distribution $P_\lambda(k) = \frac{\lambda^k}{k!}e^{-\lambda}$ can be understood as the limit of the binomial distribution in the case of "rare events".

(a) Let $p = \lambda/N$ and take $N \to \infty$ while keeping λ and k constant. Show that in this limit we can approximate $(1-p)^{N-k} \approx e^{-\lambda}$. (1P)

(b) Show similarly that $\binom{N}{k} \approx \frac{N^k}{k!}$. (1P)

(c) Use (a) and (b) to show that in this limit the binomial distribution tends to the Poisson distribution. (1P)

(d) Check that the Poisson distribution is properly normalized. (1P)

(e) Compute the moment- and cumulant-generating functions. (1P)

(f) Determine all cumulants. (1P)

Sample Solution

(a) Let us consider the logarithm of the left-hand side $(N-k)\ln(1-p)$. Since $p \to 0$ for $N \to \infty$ we can approximate

$$\ln(1-p) \approx -p + \mathcal{O}(p^2).$$

Therefore $(N-k)\ln(1-p) \approx -Np+kp \approx -Np = -\lambda$, hence $(1-p)^{N-k} \approx e^{-\lambda}$

(b) For this we have to show that $N!/(N-k)! \approx N^k$. The take the logarithm and apply Stirlings formula

$$\ln N! - \ln(N-k)! \;\approx\; N\ln N - N - (N-k)\ln(N-k) + (n-k)$$

Since $k \ll N$ we can further approximate $\ln(N-k) \approx \ln N - k/N$. Inserting this approximation in the formula given above and simplifying the expression we arrive at

$$\ln N! - \ln(N-k)! \approx k\ln N.$$

Exponentiating this we finally obtain $\binom{N}{k} \approx \frac{N^k}{k!}$.

(c) Simply insert the results from (a) and (b):

$$\binom{N}{k} p^k (1-p)^{(N-k)} \approx \frac{N^k}{k!} p^k e^{-\lambda} = \frac{\lambda^k e^{-\lambda}}{k!}$$

(d) Check normalization:

$$\sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} \;=\; e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!} \;=\; e^{-\lambda}e^{+\lambda} \;=\; 1.$$

(e) The moment-generating function can also be found in the lecture notes:

$$M(t) \;=\; \langle e^{kt}\rangle \;=\; \sum_{k=0}^{\infty}\frac{\lambda^k}{k!}e^{-\lambda}e^{kt} \;=\; e^{-\lambda}\sum_{k=0}^{\infty}\frac{(\lambda e^t)^k}{k!} \;=\; e^{\lambda(e^t-1)}.$$

$$K(t) = \ln M(t) = \lambda(e^t - 1).$$

(f) Take the derivative of the cumulant-generating function:

$$\kappa_n \;=\; \frac{\mathrm{d}^n}{\mathrm{d}t^n} K(t)\Big|_{t=0} \;=\; \lambda e^t\Big|_{t=0} \;=\; \lambda.$$

Therefore, the Poisson distribution is special in so far as all cumulants coincide.

Exercise 4.2: Central Limit Theorem (CLT) (6P)

Let $X_1, X_2, \dots, X_N$ be statistically independent and identically distributed random variables with the probability density function

$$p_X(x) = \begin{cases} e^{-x} & \text{if } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

With this exercise we would like to demonstrate the CLT for $Z_N = \sum_{i=1}^N X_i$ in the limit $N \to \infty$.

(a) Show that the N-fold convolution product of $p(x)$ is given by (1P)

$$p_{Z_N}(z) \;=\; p_X^{*N}(z) \;=\; \big(\underbrace{p_X * p_X * \cdots * p_X}_{N \text{ times}}\big)(z) \;=\; \frac{z^{N-1}}{(N-1)!}\, e^{-z} \qquad (z \geq 0)\,.$$

(b) Check the norm and compute the mean and the variance of $p_{Z_N}(z)$. (1P)

(c) Standardize the distribution by shifting and rescaling Z_N in such a way that the new random variable $\tilde{Z}_N$ has a probability density $p_{\tilde{Z}_N}(z)$ with unit norm, zero mean, and unit variance (see figure). (1P)

(d) Show that the cumulant-generating function $K_{\tilde{Z}}(t)$ of the standardized probability density $p_{\tilde{Z}_N}(z)$ is given by (1P)

$$K_{\tilde{Z}}(t) \;=\; \frac{1}{2} N \ln N - t\sqrt{N} - N \ln\left(\sqrt{N} - t\right)\,.$$

(e) Compute all cumulants κ_n and show that only κ_2 survives in the limit $N \to \infty$. What does it mean? (2P)

Sample Solution

(a) Since $p_{Z_1}(z) = p_X(z)$ the assertion can be proven recursively by showing that $p_{Z_N} * p_X = p_{Z_{N+1}}$. In fact, for $z \geq 0$ we have

$$\Big(p_{Z_N} * p_X\Big)(z) = \int_{-\infty}^{+\infty} p_{Z_N}(z') p_X(z-z')\,\mathrm{d}z' \;=\; \frac{z^N}{N!} e^{-z} \;=\; p_{Z_{N+1}}(z)\,.$$

(b) The moments are

$$m_n = \int_0^\infty \mathrm{d}z\, z^n\, p_{Z_N}(z) = \frac{\Gamma(N+n)}{\Gamma(N)} = \frac{(N+n-1)!}{(N-1)!}\,,$$

giving

$$m_0 = 1, \quad m_1 = N, \quad m_2 = N(N+1), \quad \sigma^2 = \mathrm{Var}(Z_N) = m_2 - m_1^2 = N\,.$$

(c) First we shift the random variable to the center by replacing $Z_N \to Z_N^{cent} = Z_N - N$. For the distribution function this requires to shift its argument z in opposite direction, namely $z \to z + N$:

$$p_{Z_N^{cent}}(z) \;=\; \begin{cases} \frac{(z+N)^{N-1}}{(N-1)!}\, e^{-z-N} & \text{for } z \geq -N \\ 0 & \text{otherwise.} \end{cases}$$

Next we have to rescale the random variable by $Z_N^{cent} \to \tilde{Z}_N = Z_N^{cent}/\sigma$ in order to get a new random variable $\tilde{Z}_N$ with unit variance. For the distribution function this means to scale reciprocally by $z \to z\sigma$. In addition, the total function has to be multiplied by σ in order to restore normalization:

$$p_{\tilde{Z}_N}(z) \;=\; \begin{cases} \sqrt{N}\frac{(z\sqrt{N}+N)^{N-1}}{(N-1)!}\, e^{-z\sqrt{N}-N} & \text{for } z \geq -\sqrt{N} \\ 0 & \text{otherwise.} \end{cases}$$

(d) Use e.g. *Mathematica*®to compute the moment-generating function

$$M_{\tilde{Z}_N}(t) \;=\; \langle e^{tz} \rangle \;=\; \int_{-\sqrt{N}}^{\infty} \mathrm{d}z\, e^{tz}\, p_{\tilde{Z}_N}(z) \;=\; N^{\frac{N-1}{2}+\frac{1}{2}} e^{-\sqrt{N}t} \left(\sqrt{N}-t\right)^{-N}.$$

The corresponding cumulant-generating function reads

$$K_{\tilde{Z}_N}(t) \;=\; \ln M_{\tilde{Z}_N}(t) \;=\; -t\sqrt{N} - N\log\left(\sqrt{N}-t\right) + \frac{1}{2}N\ln(N).$$

(e) In order to compute the cumulants κ_n we have to calculate the n-th derivative of $K_{\tilde{Z}_N}(t)$ with respect to t and then setting $t = 0$. This yields the following results:

$$\begin{aligned} K'_{\tilde{Z}_N}(t) &= \frac{t\sqrt{N}}{\sqrt{N}-t} && \Rightarrow && \kappa_1 = 0 \\ K''_{\tilde{Z}_N}(t) &= \frac{\sqrt{N}}{(\sqrt{N}-t)^2} && \Rightarrow && \kappa_2 = 1 \\ K^{(n)}_{\tilde{Z}_N}(t) &= \frac{N(n-1)!}{(\sqrt{N}-t)^n} && \Rightarrow && \kappa_n = \frac{(n-1)!}{N^{n/2-1}} \quad (n \geq 2) \end{aligned}$$

In the limit $N \to \infty$ the only cumulant which survives is the variance $\kappa_2 = 1$. Therefore, the stanardized distribution converges to a well-defined distribution with a variance $\sigma^2 = 1$ but no other properties. In the lecture we have shown that this is the normal distribution.

(Σ = 12P)