Exercise 1.1: Fixed points of a differential equation (3P)

(a) Determine the fixed points of the differential equation $\dot{x}(t) = ax(t)\big(1 - x^2(t)\big)$ and their stability. (1P)

(b) Let $|x(0)| \ll 1$ be very small. Compute the typical time that the system needs to 'switch' to a stable fixed point. (2P)

Sample Solution

(a) The fixed points are given by $\dot{x} = 0$, hence we get three fixed points $x^* = 0$ and $x^* = \pm 1$. (1P) For the stability we compute $\dot{x}$ in the vicinity of the fixed point $x^* + \epsilon$:

$$\dot{x} = a(x^* + \epsilon)\big(1 - (x^* + \epsilon)^2\big) = \underbrace{x^*(1 - (x^*)^2)}_{=0} + \epsilon\, \underbrace{a\,(1 - 3(x^*)^2)}_{=\gamma} + \mathcal{O}(\epsilon^2)$$

or, in short, $\dot{x} \approx \gamma\epsilon$. A fixed point x^* is stable if $\gamma < 0$. Hence for $a > 0$ both fixed points $x^* = \pm 1$ are stable while $x^* = 0$ is unstable. For $a < 0$ it is the other way round. (1P)

(b) We could solve the differential equation, obtaining the solution

$$x(t) = \pm \frac{1}{\sqrt{1 - Ce^{-2at}}}$$

where C is an integration constant determined by the initial condition:

$$x(0) = \pm \frac{1}{\sqrt{1 - C}} \quad \Rightarrow \quad C = \frac{1}{x(0)^2} - 1.$$

The typical switching time t_f is reached when $x(t)$ made 1/2 of its way towards the stable fixed point, i.e. $x(t_f) = \pm\frac{1}{2}$:

$$t_f = \frac{\log\left(\frac{1}{3}\left(\frac{1}{\mathrm{x0}^2} - 1\right)\right)}{2a} \approx const - \frac{\ln x(0)}{a}.$$

(1P)

Correction advice: In the result we would like to see $t_f \propto \ln(1/x(0))$ and $t_f \propto 1/a$ with the correct sign. The prefector and the definition 'where' exactly the flip takes place is irrelevant.

Exercise 1.2: Attractor of a quadratic map (4P)

Let us consider the nonlinear map $x_{n+1} = f(x_n)$ with $f(x) = a - x^2$, where $a > 0$ is a constant.

(a) Determine the real-valued fixed points $x^* = f(x^*)$ and assess their stability. (1P)

(b) Set $a := 1$ and $x_0 := 0.5$ and iterate the map numerically up to $n = 20$. What happens for $n \to \infty$ and how do you interpret the result? (1P)

(c) Prove your observation in (b) analytically. (2P)

Sample Solution

(a) Solving $x^* = a - (x^*)^2$ we find (for $a > 0$) two real-valued the fixed points

$$x^*_\pm = \frac{1}{2}\left(\pm\sqrt{1+4a} - 1\right).$$

In order to check the stability we compute

$$f'(x^*_\pm) = 2x^*_\pm = \pm\sqrt{1+4a} - 1$$

A fixed point is stable if $|f'(x)| < 1$. Therefore x^*_+ is stable for $a < \frac{3}{4}$, marginal at $a = \frac{3}{4}$, and unstable for $a > \frac{3}{4}$. The other fixed point x^*_- is always unstable. (1P)

(b) For $a = 1$ and $x_0 = 0.5$ the iteration sequence reads (can be computed even with a pocket calculator): 0.5 | 0.75 | 0.4375 | 0.808594 | 0.346176 | 0.880162 | 0.225315 | 0.949233 | 0.0989562 | 0.990208 | 0.0194888 | 0.99962 | 0.00075948 | 0.999999 | $1.15362 * 10^{-6}$ | 1. | $2.66165 * 10^{-12}$ | 1. | 0. | 1. | 0. | ...

That is, until sufficiently many iterations, the sequence effectively alternates between 0 and 1. This phenomenon is called *period doubling.* (1P)

(c) Apparently 0 and 1 are fixed points of the two-fold nested map

$$x_{n+2} = f(f(x_n)) = 2x^2 - x^4$$

which has four fixed points

$$x^*_0 = 0\,, \quad x^*_1 = 1\,, \quad x^*_{2,3} = -\frac{1}{2}\left(1 \pm \sqrt{5}\right).$$

Checking again $|[f(f(x))]'| - |4x - 4x^3|$ at these fixed points shows that the first two are stable while $x^*_{2,3}$ is unstable. Therefore, the sequence $0, 1, 0, 0, 1, \ldots$ is an attractive (period-doubled) fixed point. (2P)

Note: Increasing a further you will find that at $a \approx 1.25$ the fixed points $x^*_{0,1}$ become unstable and that the system bifurcates into period-quadrupling fixed points. This process of period-doubling continues until the system becomes chaotic.

Exercise 1.3: Probability density of the logistic map (5P)

Let us consider the logistic map

$$x_{n+1} = f(x_n) \qquad \text{with} \qquad f(x) = 4x(1-x)$$

The purpose of this exercise is to show that this map is fully chaotic and that the probability to find x in a certain infinitesimal interval can be computed analytically.

(a) Prove that the Dirac delta function obeys the relation $\delta(ax) = \frac{1}{|a|}\delta(x)$. (1P)

(b) Likewise prove that $\delta(F(x)) = \sum_i \frac{\delta(x-x_i)}{|F'(x_i)|}$, where the x_i are the (non-degenerate) zeros of the differentiable function F, i.e., $F(x_i) = 0$. (1P)

(c) Let $p_n(x_n)$ be the probability density of x_n, which means that $p_n(x_n)\,\mathrm{d}x_n$ is the probability to find x_n in the infinitesimal interval $[x_n, x_n + \mathrm{d}x_n]$. Later in this lecture we will show that probability density one step later is given by

$$p_{n+1}(x_{n+1}) = \int_{-\infty}^{+\infty} \mathrm{d}x_n\ \delta\Big(x_{n+1} - f(x_n)\Big)\, p_n(x_n).$$

Insert $f(x)$ into this expression and use (b) to simplify it. (2P)

(d) A probability density of a map is called *stationary* if it does not change under iteration, i.e. $p_{n+1}(x) = p_n(x)$. Show that the stationary probability density

$$p_n(x) \propto \frac{1}{\sqrt{x(1-x)}}$$

is a stationary solution. (1P)

Sample Solution

(a) The dirac delta function is a distribution that has to be placed into an integral together with a test function. The relation can be proven by substituting $|a|x = y$, implying that $\mathrm{d}x = \mathrm{d}y/|a|$:

$$\int_{-\infty}^{+\infty} \mathrm{d}x\, \delta(ax) g(x) = \int_{-\infty}^{+\infty} \mathrm{d}x\, \delta(|a|x) g(x) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}y}{|a|} \delta(y) g(y/|a|) = \frac{g(0)}{|a|}.$$

On the other hand

$$\int_{-\infty}^{+\infty} \mathrm{d}x\, \frac{\delta(x)}{|a|} g(x) = \frac{g(0)}{|a|}$$

gives the same result for any test function $g(x)$, hence we can identify the red-colored parts of the integrands.

(b) The function F is assumed to have non-degenerate zeros x_i, that is, $F(x_i) = 0$ and $F'(x_i) \neq 0$ (the x-axis is crossed but it is not touched). Clearly, $\delta(F(x))$ is non-zero only at the roots of F, i.e., at the points x_i where $F(x_i) = 0$. Since F is differentiable, we may thus Taylor-expand $F(x) = F(x_i) + F'(x_i)(x - x_i) + \mathcal{O}((x-x_i)^2)$ in the vicinity of the roots. Obviously, for each root x_i the prefactor $F'(x_i)$ play the same role as the proportionality constant a in part (a) of this exercise. Finally we have to add over all roots, giving $\delta(F(x)) = \sum_i \frac{\delta(x_i)}{|F'(x_i)|}$.

(c) We rewrite the equation as

$$p_{n+1}(x_{n+1}) = \int_{-\infty}^{+\infty} \mathrm{d}x_n\, \delta\Big(F(x_n)\Big)\, p_n(x_n),$$

where $F(x_n) := x_{n+1} - f(x_n) = x_{n+1} - 4x_n(1-x_n)$. This function has two roots $F(x_a) = F(x_b) = 0$, namely

$$x_{a,b} = \frac{1}{2}(1 \pm \sqrt{1 - x_{n+1}}).$$

In both cases the absolute value of the first derivative is given by

$$|F'(x_{a,b})| \;=\; 4\sqrt{1-x_{n+1}}.$$

Inserting these expressions we get

$$p_{n+1}(x_{n+1}) = \frac{p(x_a)+p(x_b)}{4\sqrt{1-x_{n+1}}}.$$

Note: This is a nonlocal map from the function p_n to p_{n+1}. Usually such non-local maps are very difficult to solve analytically. However, as we will see in part (d), there is a simple analytic solution for the stationary solution.

(d) We have to show that

$$p_n(x) = \frac{C}{\sqrt{x(1-x)}} \stackrel{?}{=} p_{n+1}(x),$$

where C is some proportionality constant. With $x_{a,b} = \frac{1}{2}(1 \pm \sqrt{1-x})$ and using (c) we find

$$p_{n+1}(x) \;=\; \frac{p(x_a)+p(x_b)}{4\sqrt{1-x}} \;=\; \frac{2C/\sqrt{x}+2C/\sqrt{x}}{4\sqrt{1-x}} \;\stackrel{!}{=}\; p(x). \qquad \square$$

Therefore, this probability distribution is stationary under iteration.

(Σ = 12P)