Exercise 11.1: Statistics of misprints (3P)

In a printing company, the error probability for a typo is $q = 10^{-6}$. The typos occur completely uncorrelated and the probability is the same for all characters.

(a) Estimate the numerical probability to have less than 5 typos in a book with 10^7 characters. (1P)

(b) Let n be the number of correct characters between two subsequent typos. Compute the probability distribution $P(n)$ in an infinitely long text. (1P)

(c) Determine the mean value and the variance of this distribution. (1P)

Exercise 11.2: Ultrarelativistic gas (9P)

The energy of a relativistic particle is given by $E = \sqrt{m^2c^4 + p^2c^2}$, where c is the velocity of light and m is the rest mass. A gas is called *ultrarelativistic* if the energy of the particles is so high that the rest mass can be neglected. In the following let us consider an ultrarelativistic gas with N particles in a container with the volume V:

(a) Show that the phase space volume $\Phi = \int_{E<E_0} \prod_{i=1}^N \mathrm{d}^3q_i\, \mathrm{d}^3p_i$ over all states with the total energy $E < E_0$ can be written as

$$\Phi = (4\pi V)^N I_N(E_0)\,,$$

where I_N obeys the recursion relation (3P)

$$I_N(E) = \int_0^{E/c} \mathrm{d}p\, p^2\, I_{N-1}(E - pc).$$

(b) Verify that $I_N(E)$ is a homogeneous function and show that it can be written as $I_N(E) = (E/c)^{3N} C_N$, where C_N is a (N-dependent) constant. (1P)

(c) Insert the homogeneity obtained in (b) into the recursion relation given in (a) and solve the integral (e.g. with *Mathematica*®). This gives a recursion relation for the C_N which you can solve exactly. (1P)

(d) Use this result to determine the number of states in an infinitesimal energy shell $[E, E + \delta E]$. (1P)

(e) Using Stirling's formula and taking the permutation entropy of non-distinguishable quantum particles into account, show that (1P)

$$H(E, V, N) \simeq N \left[\ln\left(\frac{8\pi V E^3}{27h^3c^3N^4}\right) + 4\right] + \dots .$$

Please turn over ⇒

(f) Compute temperature and pressure and show that the equation of state is the same as the one of the non-relativistic ideal gas. (1P)

(g) Use (f) to show that the entropy can be expressed as

$$H(E,V,N) = N \ln\left(\frac{AT^3 V}{N}\right), \qquad H(T,P,N) = N \ln\left(\frac{AT^4}{P}\right)$$

where A is a constant. Use these formulas to calculate the specific heats C_V and C_P. How does $\gamma = C_P/C_V$ differ from the non-relativistic case? (1P)

(Σ = 12P)

To be handed in on Monday, January 15, 2018, at the beginning of the lecture.