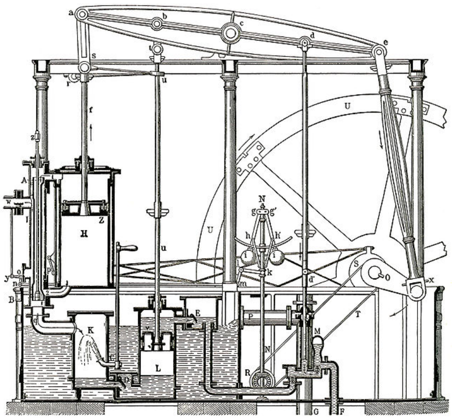
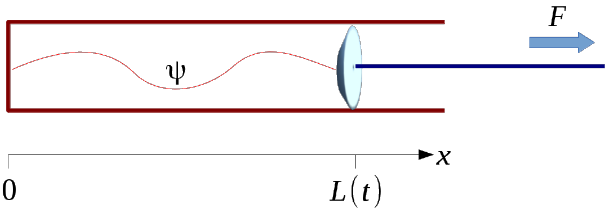


(Quantum) steam engine [Wikimedia]

Exercise 10.1: Adiabatic expansion of a quantum state (12P)



Let us consider a single quantum-mechanical particle in a cylinder with a movable piston. The system is modeled as a one-dimensional quantum system with the potential

$$V(x,t) = \begin{cases} 0 & 0 < x < L(t) \\ \infty & \text{otherwise.} \end{cases}$$

The piston is first at rest, then moves with velocity v, and finally stops again:

$$L(t) = \begin{cases} L_0 & t \le 0 \\ L_0 + vt & 0 < t < T \\ L_1 = L_0 + vT & t \ge T \end{cases}$$

The particle evolves according to the Schrödinger equation $i\hbar\partial_t\psi(x,t) = -\frac{\hbar^2}{2m}\partial_x^2\psi(x,t)$ subjected to the boundary conditions $\psi(0,t) = \psi(L(t),t) = 0$.

(a) Determine the eigenenergies E_n and eigenfunctions $\psi_n(x,t)$ for $t < 0$. (1P)

(b) Verify that

$$\phi_n(x,t) = \sqrt{\frac{2}{L(t)}}\exp\left(\frac{i\alpha x^2}{L_0 L(t)} - \frac{in^2\pi^2(L(t)-L_0)}{4\alpha L(t)}\right)\sin\left(\frac{n\pi x}{L(t)}\right)$$

for $n = 1, 2, \ldots$ and with $\alpha = \frac{mL_0 v}{2\hbar}$ solves the time-dependent Schrödinger equation in the moving phase. (2P)

(c) Prove that for $t = 0$ the non-moving states $\psi_n(x,0)$ provide an orthonormal basis. Prove the same for the moving states $\phi_n(x,0)$ as well. (2P)

Please turn over ⇒

(d) Consider the moving eigenfunction $\phi_n(x,0)$ at $t=0$ and expand it to first order in the velocity v. Let us refer to this approximated wave function as $\tilde{\phi}_n(x,0)$. (2P)

(e) Suppose that for $t<0$ the particle is in its ground state $\psi_1(x,t)$. When the piston suddenly begins to move, the particle is no longer in its ground state, i.e., the coefficients c_n in the expansion

$$|\psi_1\rangle = \sum_{n=1}^{\infty} \underbrace{\langle \tilde{\phi}_n|\psi_1\rangle}_{c_n} |\phi_n\rangle$$

do not vanish for $n>1$. This means that the particle is now in a nontrivial linear combination of all eigenmodes. To see this, compute the scalar products $\langle \tilde{\phi}_1|\psi_1\rangle$ and $\langle \tilde{\phi}_2|\psi_1\rangle$ to lowest order in v at $t=0$. (2P)

(f) In the adiabatic limit of very very small velocities the particle remains practically in its ground state ($c_1 \approx 1$ and $c_n \approx 0$ for $n>1$). Assuming that the particle is actually in the state ϕ_1 during expansion, compute the expectation value of the energy as a function of time. (2P)

(g) Calculate the energy loss $\Delta E = \langle E\rangle_{t=0} - \langle E\rangle_{t=T}$ during expansion and use it to calculate the force F that the particle exerts on the piston as well as the mechanical work ΔW performed by the piston. (1P)

(Σ = 12P)

Frohes Fest
und einen guten Start ins neue Jahr!

To be handed in on Monday, January 08, 2018, at the beginning of the lecture.