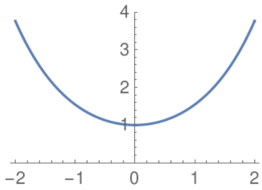
Exercise 6.1: Transforming probability densities (6P)



Consider the curve $y = \cosh x$. The aim of this exercise is to decorate this curve in the interval $x \in [x_1, x_2]$ with random points in such a way that the density of the points is uniform along the curve. Since the slope of the curve varies, this means that the x-coordinates of these points are *not* uniformly distributed.

(a) Compute the probability density $p(x)$ of the x-coordinates. Hint: the density of the points per arc length of the curve has to be constant. (2P)

(b) Normalize the probability density on the interval $x \in [x_1, x_2]$. (1P)

(c) Find a function $f : z \mapsto x = f(z)$ such that it maps a uniform probability density $p(z) = const$ to the non-uniform probability density $p(x)$ calculated in (a)-(b). (2P)

(d) Adjust the integration constant in (c) in such a way that f maps $[z_1, z_2] \mapsto [x_1, x_2]$ with $f(z_1) = x_1$ and $f(z_2) = x_2$. Specialize the result for a standard random number ($z_1 = 0, z_2 = 1$). (1P)

Exercise 6.2: Relative entropy (2P)

The relative entropy $H(p||q)$ of two probability distributions $p_1, \ldots, p_N$ and $q_1, \ldots, q_N$, which is also known as Kullback-Leibler-Divergence $D(p||q)$, measures how different the probability distributions are. It is defined by

$$H(p||q) = D(p||q) = \sum_i p_i \ln \frac{p_i}{q_i}.$$

(a) Use Jensen's inequality to show that $H(p||q) \geq 0$. (1P)

(b) Show that $H(p||p) = 0$ and $H(p||u) = \ln N - H(p)$, where u is the uniform (constant) probability distribution. (1P)

Exercise 6.3: Entropy of overlapping probability densities (4P)

Let $p(x)$ be a given continuously differentiable probability density. Let us shift it to the right and to the left by $x \to x \pm a$ and create a mixed probability density of the form

$$q(x) := \frac{1}{2}\Big(p(x-a) + p(x+a)\Big).$$

The purpose of this exercise is that the entropy is minimal for $a = 0$ where both of them match, that is, the entropy is minimized in the case of a perfect overlap.

(a) Show that $\int_{-\infty}^{+\infty} \Big(p'(x+a) - p'(x-a)\Big)\, \mathrm{d}x = 0 \quad \forall a$. (1P)

(b) Prove that the entropy $H_a = -\int_{-\infty}^{+\infty} q(x) \ln q(x)\, \mathrm{d}x$ is extremal for $a = 0$. (1P)

(c) Show that the extremum at $a = 0$ is a minimum. You can assume that $p(x)$ and all its derivatives vanish at $x = \pm\infty$. (2P)

Note: Is this minimization of overlapping functions useful? Yes, it is, you can use it to tune your piano, see http://piano-tuner.org. Available for Windows, Mac, Android, and iOS.

(Σ = 12P)

To be handed in on Monday, November 27, at the beginning of the lecture.