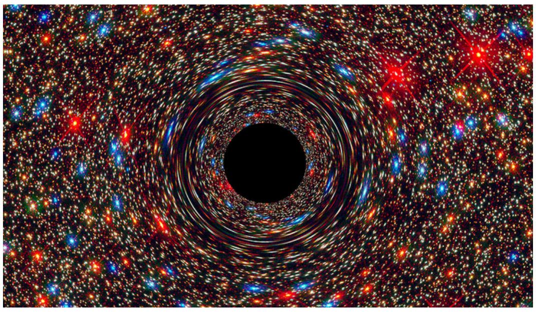


Black holes are not black [NASA CCBY 2.0]

Exercise 5.1: Playing with black holes (4P)

A (non-rotating electrically neutral) black hole is space-time singularity described by a single parameter, namely, its mass M. It is surrounded by a spherical event horizon with radius $r = 2GM/c^2$ and surface area $A = 4\pi r^2$, where G is the gravitational constant.[1]

(a) Suppose that a black hole has a certain entropy H depending on M. Compute the black hole temperature T for given $H(M)$ using the Clausius relation $\mathrm{d}E = T\,\mathrm{d}H$ and Einsteins $E = Mc^2$. (1P)

(b) A black hole with temperature $T > 0$ is not black: it is expected to emit radiation like a black body. According to Wien's displacement law we expect a typical wave length $\lambda \approx \hbar c/4k_B T$. Since the black hole has only a single relevant length scale, namely, its horizon radius r, it is natural to conjecture that $\lambda \approx r$. Use this conjecture to compute $T(M)$ and $H(M)$. (1P)

(c) Show that $H(M)$ is proportional to the surface area of the black hole. (1P)

(d) Ordinary physical units (m,s,kg) are superfluous since ther are natural Planck units. For example, the Planck length and area are given by $l_P = \sqrt{\hbar G/c^3}$ and $A_P = l_P^2$. How many bits per unit Planck area reside on the surface of a black hole? (1P)

Exercise 5.2: Waiting for an unlikely event (4P)

Consider a random experiment which yiels the result 'A' with probability $p < 1$ and 'B' otherwise. Let X be a random variable defined as the number of trials *before* you get the first 'B' (for example, 'B'$\mapsto$ 0, 'AB'$\mapsto$ 1, 'AAB' $\mapsto$ 2, ...).

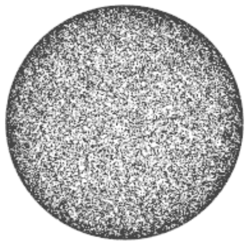
(a) Specifiy the probability distribution $P_X : \mathbb{N}_0 \to [0, 1] : \; x \mapsto P_X(x)$. (1P)

(b) Check the normalization of your result in (a). (1P)

(c) Determine the average entropy $H_X = \langle H_X(x)\rangle_x$. How many bits of information do you get on average for $p = 0.5$ and $p = 0.1$? What happens in the limits $p \to 0$ and $p \to \infty$? (2P)

[1]This exercise uses simplified assumptions. The results will differ from the correct results obtained by Hawking and Bekenstein by constant factors such as π.

Exercise 5.3: Uniform probability density on a unit sphere (4P)



Consider a unit sphere with standard spherical coordiates $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$. The figure shows uniformly distributed random points on the surface.

(a) Calculate the normalized probability density $p(\theta, \phi)$ which is uniform (i.e. constant per unit area) on the surface of the sphere. (1P)

(b) Show that this probability density factorizes. (1P)

(c) Let X_1 and X_2 be two uncorrelated continuous random variables with a uniform distribution between 0 and 1 (e.g. x_1, x_2 can be drawn from a standard random number generator). Transform $x_1, x_2 \in [0, 1]$ to spherical coordinates θ, ϕ in such a way that the resulting probability on the unit sphere is uniform. (2P)

(Σ = 12P)

To be handed in on Monday, November 20, at the beginning of the lecture.