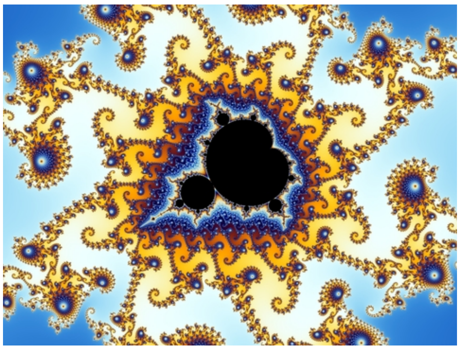

Fractal of a quadratic map in the complex plane [Wikimedia]

Exercise 1.1: Fixed points of a differential equation (3P)

(a) Determine the fixed points of the differential equation $\dot{x}(t) = ax(t)\big(1 - x^2(t)\big)$ and their stability. (1P)

(b) Let $|x(0)| \ll 1$ be very small. Compute the typical time that the system needs to 'switch' to a stable fixed point. (2P)

Exercise 1.2: Attractor of a quadratic map (4P)

Let us consider the nonlinear map $x_{n+1} = f(x_n)$ with $f(x) = a - x^2$, where $a > 0$ is a constant.

(a) Determine the real-valued fixed points $x^* = f(x^*)$ and assess their stability. (1P)

(b) Set $a := 1$ and $x_0 := 0.5$ and iterate the map numerically up to $n = 20$. What happens for $n \to \infty$ and how do you interpret the result? (1P)

(c) Prove your observation in (b) analytically. (2P)

Exercise 1.3: Probability density of the logistic map (5P)

Let us consider the logistic map

$$x_{n+1} = f(x_n) \qquad \text{with} \qquad f(x) = 4x(1-x)$$

The purpose of this exercise is to show that this map is fully chaotic and that the probability to find x in a certain infinitesimal interval can be computed analytically.

(a) Prove that the Dirac delta function obeys the relation $\delta(ax) = \frac{1}{|a|}\delta(x)$. (1P)

(b) Likewise prove that $\delta(F(x)) = \sum_i \frac{\delta(x-x_i)}{|F'(x_i)|}$, where the x_i are the (non-degenerate) zeros of the differentiable function F, i.e., $F(x_i) = 0$. (1P)

(c) Let $p_n(x_n)$ be the probability density of x_n, which means that $p_n(x_n)\,\mathrm{d}x_n$ is the probability to find x_n in the infinitesimal interval $[x_n, x_n + \mathrm{d}x_n]$. Later in this lecture we will show that probability density one step later is given by

$$p_{n+1}(x_{n+1}) = \int_{-\infty}^{+\infty} \mathrm{d}x_n \; \delta\Big(x_{n+1} - f(x_n)\Big) \, p_n(x_n).$$

Insert $f(x)$ into this expression and use (b) to simplify it. (2P)

(d) A probability density of a map is called *stationary* if it does not change under iteration, i.e. $p_{n+1}(x) = p_n(x)$. Show that the stationary probability density

$$p_n(x) \propto \frac{1}{\sqrt{x(1-x)}}$$

is a stationary solution. (1P)

(Σ = 12P)